

INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Chain rule

LECTURE

10.1. In order to take the derivative of a composition of functions like $f(x) = \sin(x^7)$, the product rule does not work. The functions are not multiplied but are “chained” in the sense that we evaluate first x^7 then apply the sin function to it. In order to differentiate, we take the derivative of the x^7 then multiply this with the derivative of the function sin evaluated at x^7 . The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) .$$

10.2.

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} .$$

Write $H(x) = g(x+h) - g(x)$ in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} .$$

As $h \rightarrow 0$, we also have $H \rightarrow 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit.

10.3. Let us look at some examples.

Example: Find the derivative of $f(x) = (4x^2 - 1)^{17}$. **Solution** The inner function is $g(x) = 4x^2 - 1$ with derivative $8x$. We get therefore $f'(x) = 17(4x - 1)^6 \cdot 8x$. **Remark.** We could have expanded out the power $(4x^2 - 1)^{17}$ first and avoided the chain rule. Try it. You will see that the rule of avoiding the **chain rule** is called the **pain rule**.

Example: Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution:** applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

Example: For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac . This agrees with the definition of f times the derivative of g . You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

10.4. One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

Example: Find the derivative of the natural logarithm function $\log(x)$.¹ **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x)) \log'(x) = x \log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$$\frac{d}{dx} \log(x) = 1/x.$$

Definition: Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

Example: Find the derivative of $\arcsin(x)$. **Solution.** We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Example: Find the derivative of $\arccos(x)$. **Solution.** We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Example: $f(x) = \sin(x^2 + 3)$. Then $f'(x) = \cos(x^2 + 3)2x$.

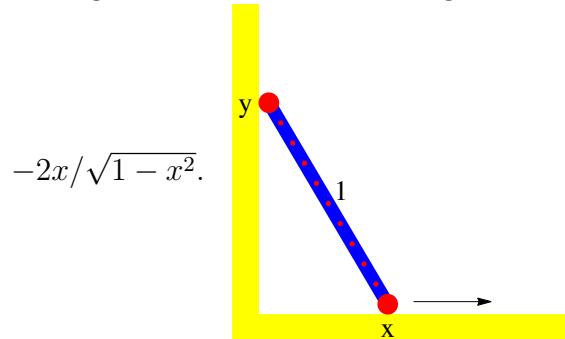
Example: $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x)$. Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

Example: Let us compute the derivative of $\sin(\sqrt{x^5 - 1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5 - 1}$ gives $\cos(\sqrt{x^5 - 1}) \frac{d}{dx} \sqrt{x^5 - 1}$. Apply now the chain rule again for the derivative on the right hand side.

¹We always write $\log(x)$ for the natural log. Similarly as $\exp(x) = e^x$, one can also use \ln which stands for "logarithmus naturalis". Practically all computer languages like Python, C, Perl, R, Matlab, Mathematica use \log . Paul Halmos called "ln" a childish notation which no mathematician ever used.

Example: Here is a famous **falling ladder problem**.

A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x . We have $y = f(x) = \sqrt{1 - x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) =$



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

Example: For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))$)).

Example: Take the derivative of $f_3(x) = e^{e^x}$.

Solution We can also write this as $\exp(\exp(\exp(x)))$. The derivative is

$$\exp(\exp(\exp(x))) \exp(\exp(x)) \exp(x) .$$

Example: Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))$$

Here is the poetic formula obtained when running this in Mathematica:

$$F[f.] := \text{Exp}[f]; \text{D}[\text{Last}[\text{NestList}[F, x, 11]], x]$$

$$\exp \left(e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{e^x}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^x}} + e^{e^x} + e^x + x \right)$$

Example: Find the derivative of $1/\sin(x)$ using the quotient rule.

Solution $-\cos(x) \cdot 1/\sin^2(x)$.

Example: Find the derivative of $f(x) = 1/\sin(x)$ using the chain rule.

Solution. The outer function is $f(x) = 1/x$. Therefore $f'(x) = -\cos(x)/\sin^2(x)$.

Homework

Problem 10.1: Find the derivatives of the following functions:

a) $f(x) = \log(\log(x))$

c) $f(x) = x/(4 + x^2)$

b) $f(x) = \cot(x^{17})$

d) $(\sin(x) + \cos(x))^{-3}$

Problem 10.2: Find the derivatives of the following functions at $x = 1$.

a) $\sqrt{x^2 - 1}$

b) $f(x) = -x \log(x)$. (where \log is natural log)

c) $(1 + x^3)^{100}$

log)

d) $\sin(\sin(\sin(x)))$

Problem 10.3: a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$ and using the product rule.

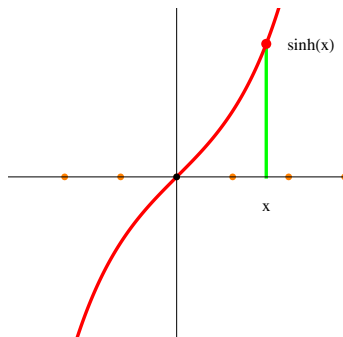
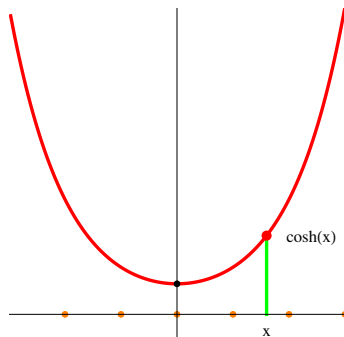
b) Find the derivative of $f(x) = \operatorname{arccot}(x)$ by differentiating $\cot(\operatorname{arccot}(x)) = x$ and using the chain rule.

Problem 10.4: a) Find the derivative of \sqrt{x} by differentiating the identity $f(x)^2 = x$ leaving f as it is and solving for $f'(x)$.

b) Find the derivative of $x^{m/n}$ by differentiating the identity $f(x)^n = x^m$ leaving f as it is and solving for $f'(x)$.

Problem 10.5: a) Find the derivative of the inverse $\operatorname{arccosh}(x)$ of $\cosh(x)$ by using the chain rule.

b) Find the derivative of the inverse $\operatorname{arsinh}(x)$ of $\sinh(x)$ by using the chain rule.



Define $\cosh(x) = [\exp(x) + \exp(-x)]/2$ and $\sinh(x) = [\exp(x) - \exp(-x)]/2$. the **hyperbolic cosine** and **hyperbolic sine**. The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine. You check $\cosh^2(x) - \sinh^2(x) = 1$.