INTRODUCTION TO CALCULUS

MATH 1A

Unit 7: Rate of Change

Lecture

7.1. Given a function f and a constant $h > 0$, we can look at the new function

$$Df(x) = \frac{f(x+h) - f(x)}{h} .$$

It is the **average rate of change** of the function with **step size** h. When changing x to $x+h$ and then $f(x)$ changes to $f(x+h)$. The quotient $Df(x)$ is a **slope** and **"rise over run"**. In this lecture, we take the limit $h \to 0$. It is called the **instantaneous rate of change**. We derive the important formulas $\frac{d}{dx} x^n = nx^{n-1}$, $\frac{d}{dx} \exp(ax) = a\exp(ax)$, $\frac{d}{dx}\sin(ax) = a\cos(ax)$, $\frac{d}{dx}\cos(ax) = -a\sin(ax)$ which we have seen already before in a discrete setting. But now we see them also to in the limit $h \to 0$:

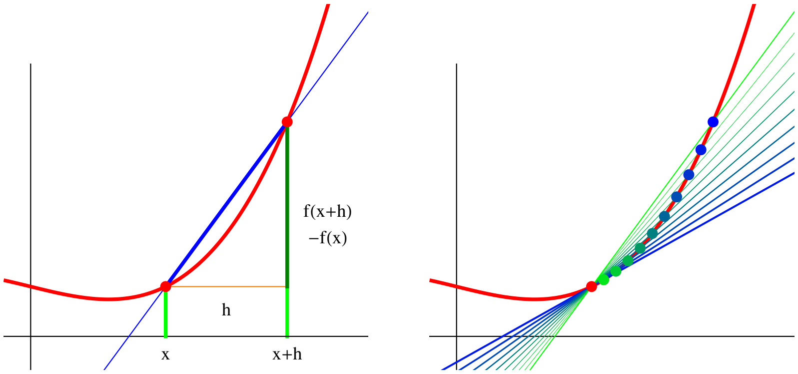
> **Definition:** If the limit $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exist, we say f is **differentiable** at the point x. The value is called the **derivative** or **instantaneous rate of change** of the function f at x. We denote the limit also with $f'(x)$.

7.2. Example. For $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

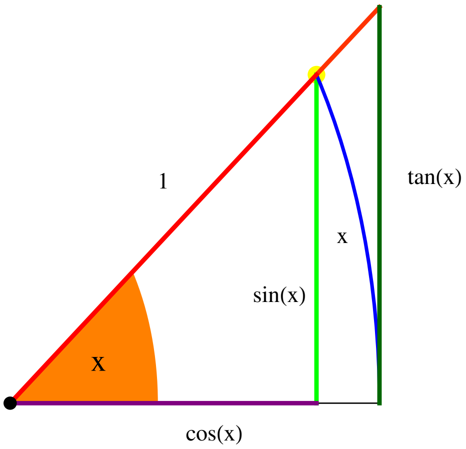
Dividing this by h gives $-2x - h$. The limit $h \to 0$ gives $-2x$. We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$. For $x = 3$, this is -6.
Example. For $f(x) = |x|$, we have $(f(x+h) - f(x))/h = 1$ if $x > 0$ and $(f(x+h) - f(x))/h = -1$ if x is smaller than $-h$. The limit $h \to 0$ does not exist at $x = 0$!

The derivative $f'(x)$ has a geometric meaning. It is the slope of the tangent at x. This is an important geometric interpretation. Often, x is "time" and the derivative as the rate of change of the quantity $f(x)$ in time.



For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

Proof: We can use our discrete calculus set-up and note that $[x]^n$ goes to x^n for $h \to 0$. More traditional is an expansion $f(x+h)-f(x) = (x+h)^n = (x^n+nx^{n-1}h+a_2h^2+\cdots+h^n)-x^n = nx^{n-1}h+a_2h^2+...+h^n)$. If we divide by h, we get $nx^{n-1}+h(a_2+\cdots+h^{n-2})$ for which the limit $h \to 0$ exists: it is nx^{n-1}. This example is important because many functions can be approximated very well with polynomials.



For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0+h)-f(0)]/h = \sin(h)/h = \mathrm{sinc}(h)$. We have already mentioned that the limit is 1 before. Lets look at it again geometrically. For all $0 < x < \pi/2$ we have

$$\sin(x) \leq x \leq \tan(x) \; .$$

Now divide everything by $\sin(x)$. Because $\tan(x)/\sin(x) = 1/\cos(x) \to 1$ for $x \to 0$, the value of $\mathrm{sinc}(x) = \sin(x)/x$ must go to 1 as $x \to 0$. Renaming the variable x with the variable h, we have verified the **fundamental theorem of trigonometry**

$$\boxed{\lim_{h \to 0} \frac{\sin(h)}{h} = 1}$$

7.3. For $f(x) = \cos(x)$ we have $f'(0) = 0$. To see this, look at $[f(0+h)-f(0)]/h = [\cos(h)-1]/h$. From $2-2\cos(h) = \sin^2(h)+(1-\cos(h))^2$ which is less than h^2 (geometry!) we have $(1-\cos(h) \leq h^2/2$ so that $(1-\cos(h))/h \leq h/2$. We have now

$$\boxed{\lim_{h \to 0} \frac{(1-\cos(h)}{h} = 0}.$$

The interpretation is that the tangent is **horizontal** for the cos function at $x = 0$.

7.4. From the previous two examples and trig identities we get

$$\cos(x+h)-\cos(x) = \cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x) = \cos(x)(\cos(h)-1)-\sin(x)\sin(h) \; .$$

Now use the just established $(\cos(h)-1)/h \to 0$ and the fundamental theorem of trigonometry $\sin(h)/h \to 1$ to see that $[\cos(x+h)-\cos(x)]/h \to -\sin(x)$.

For $f(x) = \cos(ax)$ we have $f'(x) = -a\sin(ax)$.

7.5. Similarly,

$$\sin(x+h)-\sin(x) = \cos(x)\sin(h)+\sin(x)\cos(h)-\sin(x) = \sin(x)(\cos(h)-1)+\cos(x)\sin(h)$$

because $(\cos(h)-1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\sin(x+h)-\sin(x)]/h \to \cos(x)$.

for $f(x) = \sin(ax)$, we have $f'(x) = a\cos(ax)$.

$$e^x = \lim_{n \to \infty} (1+\frac{1}{n})^{nx}$$

7.6. Like π, the **Euler number** $e = e^1$ is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large n, like $n = 100$ and compute $(1 + 1/n)^n$. For $n = 100$ for example, we see $101^{100}/100^{100}$. We only need to compute the integer 101^{100} and then put a comma after the first digit to get a decent approximation of e.

7.7. To see why the limit exists, verify hat the fractions $A_n = (1 + 1/n)^n$ increase and $B_n = (1 + 1/n)^{(n+1)}$ decrease. Since $B_n/A_n = (1 + 1/n)$ which goes to 1 for $n \to \infty$, the limit exists. The same argument shows that $(1 + 1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1 + 1/n)$ decreases. The limiting function $\exp(x) = e^x$ is called the **exponential function**. Remember that if we write $h = 1/n$, then $(1 + 1/n)^{nx} = \exp_h(x)$ considered earlier in the course. We can sandwich the exponential function between $\exp_h(x)$ and $(1 + h)\exp_h(x)$:

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1 + h), \quad x \geq 0 .$$

For $x < 0$, the inequalities are reversed.

7.8. Lets compute the derivative of $f(x) = e^x$ at $x = 0$. **Answer**. We have for $x \leq 1$

$$1 \leq (e^x - 1)/x \leq 1 + x .$$

Therefore $f'(0) = 1$. The exponential function has a graph which has slope 1 at $x = 0$. Now, we can get the general case. It follows from $e^{x+h} - e^x = e^x(e^h - 1)$ that the derivative of $\exp(x)$ is $\exp(x)$.

For $f(x) = \exp(ax)$, we have $f'(x) = a\exp(ax)$.

7.9. It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$. We also have $(af(x))' = af'(x)$. From this, we can now compute many derivatives

7.10. Find the slope of the tangent of $f(x) = \sin(3x) + 5\cos(10x) + e^{5x}$ at the point $x = 0$. **Solution**: $f'(x) = 3\cos(3x) - 50\sin(10x) + 5e^{5x}$. Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.
Finally, lets mention an example of a function which is not everywhere differentiable.

7.11. The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x =< 1$. The derivative does not exist at $x = 0$ even so the function is continuous there. You see that the slope of the graph jumps discontinuously at the point $x = 0$.

7.12. For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$. Remember these bad guys?

To the end, you might have noticed that in the boxes, more general results have appeared, where x is replaced by ax. We will look at this again but in general, the relation $f'(ax) = af(ax)$ holds ("if you drive twice as fast, you climb twice as fast").

Homework

Problem 7.1: For which of the following functions does the derivative $f'(x)$ exist for every x?

a) $\sin(19x)$
b) $|19\exp(3x)|$
c) $4x + \exp(7x) + 3\sin(45x)$
d) $|\sin(8x)|$
e) $\sin(7/x)$
f) $\exp(-x) + |1 + \cos(15x)|$

Problem 7.2: a) A circle of radius r encloses a disc of area $f(r) = \pi r^2$. Find $\frac{d}{dr}f(r)$. Evaluate the rate of change at $r = 1/10$.
b) The ball of radius r has the volume $f(r) = 4\pi r^3/3$. Find $\frac{d}{dr}f(r))$ at $r = 1/10$ and compare it with the surface area of the sphere bounding the ball.
c) A **hypersphere** of radius r has the **hyper volume** $f(r) = \pi^2 r^4/2$. Find $\frac{d}{dr}f(r)$ and evaluate it at $r = 1/10$.

Problem 7.3: Find the derivatives of the following functions at the point $x = 0$.
a) $f(x) = 7\exp(7x) + 3\sin(22x) + x^7 + 7^x$.
b) $f(x) = 4(x^7 - 1)/(x - 1) + \cos(22x)$. First heal this function.
c) $f(x) = \frac{1+5x+10x^2+10x^3+5x^4+x^5}{x^2+2x+1}$. Also here, first heal!

Problem 7.4: In this problem we compute the derivative of $x^{1/3}$ for $x > 0$. To do so, we have to find the limit

$$\lim_{h \to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} .$$

Hint: as done in lesson 6 for $\sqrt{x}$ multiply and divide with something and use algebra in particular $(a - b)(a^2 + a * b + b^2) = a^3 - b^3$.

Problem 7.5: Find the derivative of $f(x) = (x+1)^{100} + 100^{x+1}$ at $x = 0$ without actually multiplying out the power. Argue first why $d/dx(x + 1)^n = n(x + 1)^{n-1}$ in general and why $d/dx b^{ax+c} = \log(b)b^{ax+c}$.