INTRODUCTION TO CALCULUS

MATH 1A

Unit 6: Fundamental theorem

Lecture

6.1. **Calculus** is a theory of **differentiation** and **integration**. We explore here this concept again in a simple setup and practice differentiation and integration **without taking limits**. We fix a positive constant h and take differences and sums. The fundamental theorem of calculus for $h = 1$ generalizes. We can then differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with actual derivatives and integrals. But now, we can work with arbitrary **continuous functions**. The constant h never pops up. You can think of it as something fixed, like the God-given **Planck constant** $1.6 \cdot 10^{-35}m$. In the standard calculus of Newton and Leibniz the limit $h \to 0$ is taken.

Definition: Given $f(x)$, define the **difference quotient**

$$\boxed{Df(x) = \frac{f(x+h)-f(x)}{h}}$$

6.2. If f is continuous then Df is a continuous. For simplicity, we call it "derivative". We keep the positive constant h fixed. As an example, let us take the **constant function** $f(x) = 5$. We get $Df(x) = (f(x+h) - f(x))/h = (5-5)/h = 0$ everywhere. You see that in general, if f is a constant function, then $Df(x) = 0$.

6.3. $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if $f(x) = mx + b$, then $Df(x) = \boxed{m}$.

For $f(x) = c$ we have $Df(x) = 0$. For $f(x) = mx + b$, we have $Df(x) = m$.

6.4. For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h = \boxed{2x + h}$.

6.5. For $f(x) = \sqrt{x}$ we compute $Df(x) = (\sqrt{x+h} - \sqrt{x})/h = \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$ which is $1/(\sqrt{x+h} + \sqrt{x})$. For $h \to 0$, we get $1/(2\sqrt{x})$.

6.6. Given a function f, define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \le jh < x$ with $x = nh$.

Definition: Given $f(x)$ define the **Riemann sum**

$$\boxed{Sf(x) = h[\, f(0)+f(h)+f(2h)+\cdots+f((n-1)h)\,]}$$

In short hand, we call Sf also the "integral" or "anti-derivative" of f. It will become the integral in the limit $h \to 0$ later in the course.

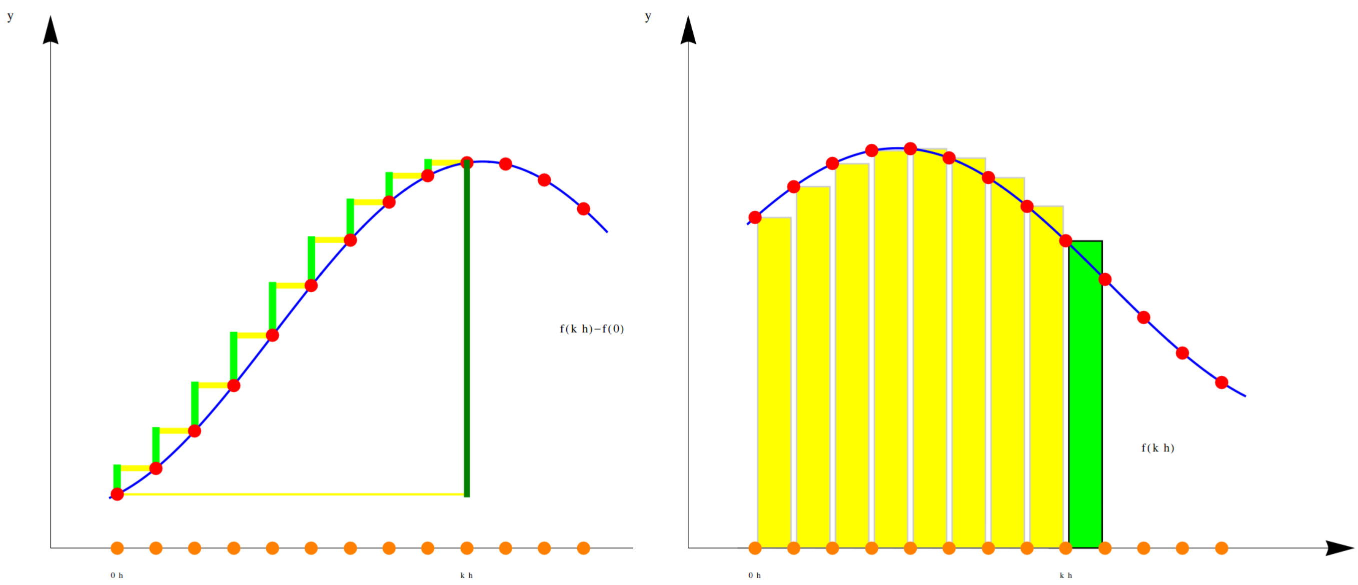
6.7. Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \le h$, and $Sf(x) = h$ for $h \le x < 2h$ and $Sf(x) = 2h$ for $2h \le x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference $Df(x)$ will become the **derivative** $f'(x)$.

The sum $Sf(x)$ will become the **integral** $\int_0^x f(t)\,dt$.

Df means rise over run and is close to the slope of the graph of f.

Sf means areas of rectangles and is close to the area under the graph of f.



6.8. Here is the **quantum fundamental theorem of calculus**

Theorem: Sum the differences gives

$$SDf(kh) = f(kh) - f(0)$$

Theorem: Difference the sum gives

$$DSf(kh) = f(kh)$$

Example: For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$$f(x+h)-f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h)-(x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $\boxed{D[x]^m = m[x]^{m-1}}$

6.9. This leads to differentiation formulas for **polynomials**. We will leave away the square brackets later to make it look like the calculus we will do later on. In the homework, we already use the usual notation.

6.10. If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$. The fundamental theorem allows us to integrate and get $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

> **Definition:** Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero.

As a rescaled exponential, it is continuous and monotone. Indeed, using rules of the logarithm we can see $\exp_h(x) = e^{x(\log(1+h)/h)} = e^{xA}$. It is actually a classical exponential with some constant A.

6.11. The function $\exp_h(x) = (1+h)^{x/h}$ has the property that its derivative is the function again (see unit 4). We also have $\exp_h(x+y) = \exp_h(x)\exp_h(y)$. More generally, define $\exp(a \cdot x) = (1+ah)^{x/h}$. It satisfies $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$ We write a dot because $\exp_h(ax)$ is not equal to $\exp_h(a \cdot x)$. For now, only the differentiation rule for this function is important.

6.12. If a is replaced with ai where $i = \sqrt{-1}$, we have $\exp(1+ia)(1+aih)^{x/h}$ and still $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Taking real and imaginary parts define new **trig functions** $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i\sin_h(a \cdot x)$. These functions are real and morph into the familiar cos and sin functions for $h \to 0$. For any $h > 0$ and any a, we have now $\boxed{D\cos_h(a \cdot x) = -a\sin_h(a \cdot x) \text{ and } D\sin_h(a \cdot x) = a\cos_h(a \cdot x).}$ We will later derive these identities for the usual trig functions.

6.13.

> **Definition:** Define $\log_h(x)$ as the inverse of $\exp_h(x)$ and
> $$1/[x+a]_h = D\log_h(x+a).$$

6.14. We have directly from the definition $\boxed{S1/[x+1]_h = \log_h(x+1)}$ As a consequence we can compute things like

$$S\frac{1}{[3x+3]} = \frac{1}{3}S\frac{1}{[x+1]} = \frac{1}{3}\log_h(x+1)\ .$$

More generally $S(1/[x+a]) = \log(x+a) - \log(a)$.

Homework

Use the differentiation and integration rules to find.

> **Problem 6.1:** Find the derivatives $Df(x)$ of the following functions:
>
> a) $f(x) = x^{111} - 3x^{14} + 5x^3 + 1$
> b) $f(x) = -x^7 + 8\log(x)$
> c) $f(x) = -3x^{13} + 17x^{5/2} - 8x$.
> d) $f(x) = \log(x+1) + 7\sqrt{x}$.

Problem 6.2: Find the integrals $Sf(x)$ of the following functions assuming $Sf(0) = 0$:

a) $f(x) = x^{10} - 8\sqrt{x}$. b) $f(x) = x^2 - 6x^7 - x$
c) $f(x) = -3x^3 + 17x^2 - 5x$ d) $f(x) = \exp(7x) + \sin(19x)$

Problem 6.3: Find the derivatives $Df(x)$ of the following functions
a) $f(x) = \exp(9 \cdot x + 3) + 2x^6$ b) $f(x) = 8\exp(-3 \cdot x) + 9x^6$
c) $f(x) = \exp(6 \cdot x) + \log(1 + x)$ d) $\log(1 - x^2)$

Problem 6.4: a) Assume $h = 1/100$. Use Wolfram alpha to plot $\cos_h(x)$ and $\sin_h(x)$ on the interval $[-2\pi, 2\pi]$. **Hint.** This means you have to plot the real and imaginary part of $(1 + i/100)^{100x}$. If you enter the expression into Wolfram alpha, it will plot the real and imaginary part.
b) Do the same for $h = 1/1000$. What has changed?

Problem 6.5: a) Write down again on your own that if $f(x) = (1 + ah)^{x/h}$, then verify $Df(x) = af(x)$. (We might have seen this already. Do it again!).
b) Write down again on your own that if $f(x) = x(x-h)(x-2h)(x-3h)$, then $Df(x) = 4x(x-h)(x-2h)$ meaning $D[x]^4 = 4[x]^3$.

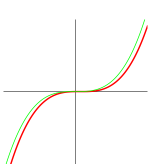
Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Differentiation rules
$Dx^n = nx^{n-1}$
$De^{a \cdot x} = ae^{a \cdot x}$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$
$D\sin(a \cdot x) = a\cos(a \cdot x)$
$D\log(1 + xa) = a/(1 + ax)$

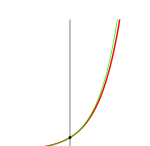
Integration rules (for $x = kh$)
$Sx^n = x^{n+1}/(n+1)$
$Se^{a \cdot x} = (e^{a \cdot x} - 1)/a$
$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$S\sin(a \cdot x) = (1 - \cos(a \cdot x))/a$
$S\frac{1}{1+ax} = \log(1 + ax)/a$

Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x+h)$.

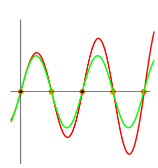
Pictures



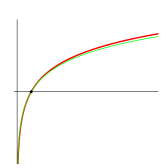
$[x]_h^3$ for $h = 0.1$



$\exp_h(x)$ for $h = 0.1$



$\sin_h(x)$ for $h = 0.1$



$\log_h(x)$ for $h = 0.1$

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021