

INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Continuity

LECTURE

4.1.

Definition: A function f is **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \rightarrow f(x_0)$ for $x \rightarrow x_0$. A function f is **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

4.2. In the interior (a, b) , the limit needs to exist both from the right and from the left. Intuitively, a function is continuous if one can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$. Some functions like $(x^2 - 1)/(x - 1)$ or $\sin(x)/x$ need to have function values filled in to become continuous.

4.3. Example. Any polynomial like x^3 or trig functions like $\cos(x)$, $\sin(x)$, $\exp(x)$ for example are continuous. Also the **sum and products** of continuous functions is continuous. For example, $x^5 + \sin(x^3 + e^x)$ is continuous everywhere. We can also **compose** continuous functions like $\exp(\sin(x))$ and still get a continuous function.

4.4. The function $f(x) = 1/x$ is continuous except at $x = 0$. It is a prototype with a **pole discontinuity** at $x = 0$. One can draw a **vertical asymptote**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere. The function can be healed at 0 even so it was at first not defined at $x = 0$.

4.5. The logarithm function $f(x) = \log|x|$ is continuous for all $x \neq 0$. It is not continuous at 0 because $f(x) \rightarrow -\infty$ for $|x| \rightarrow 0$. It might surprise you that $f(x) = (1 - x^2)/\log|x|$ can be extended to a continuous function. It is not defined at first at $x = 0$ as $\log|0| = -\infty$. It is also not defined at $x = 1$ or $x = -1$ at first because $\log(1) = 0$. But in both cases, we can heal it and see $f(1) = f(-1) = 0$. The value $f(0) = 0$ is easier to see, but filling in the value $f(1) = f(-1) = -2$ is less obvious. We will learn later to heal the function at these two points. It will need hospitalization.

4.6. The **co-secant function** $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and more generally for any multiple of π . It has poles there because $\sin(x)$ is zero at those points and because we divide by zero at such points. The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

4.7. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just pick $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

4.8. The **signum function** $f(x) = \text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

It is a prototype of a function with a **jump** discontinuity at 0. There is no way we can make this continuous at 0

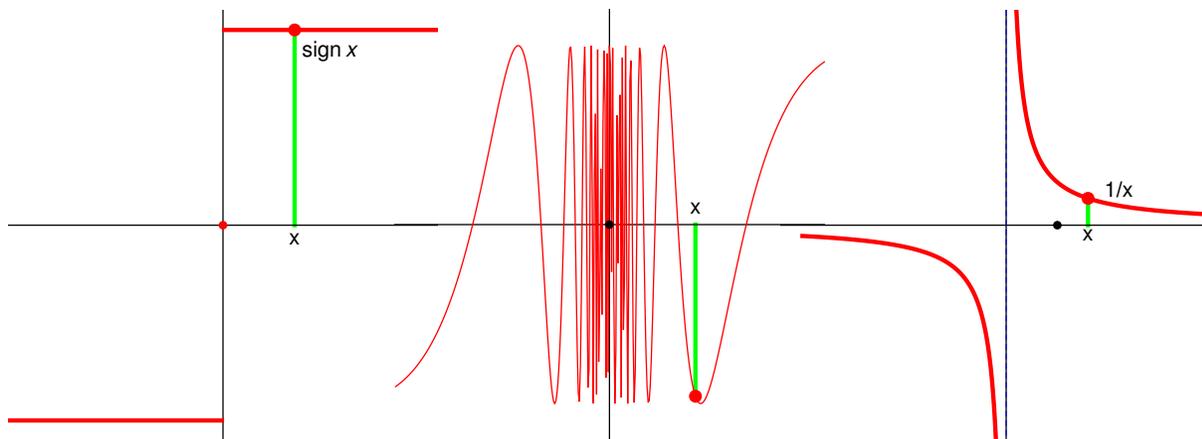
Rules:

- If f and g are continuous, then $f + g$ is continuous.
- If f and g are continuous, then $f * g$ is continuous.
- If f and g are continuous and if $g \neq 0$ everywhere, then f/g is continuous.
- If f and g are continuous, then $f \circ g(x) = f(g(x))$ is continuous.

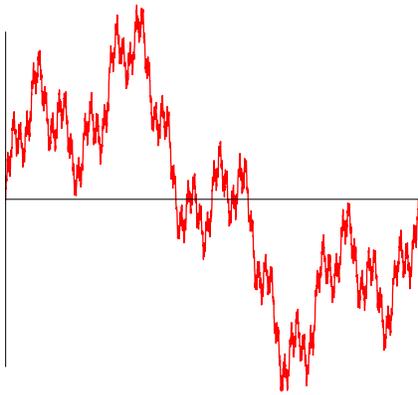
Examples. a) $f(x) = \sqrt{x^2 + 1}$ is continuous everywhere on the real line. b) $f(x) = \cos(x) + \sin(x)$ is continuous everywhere. c) $f(x) = \log(|x|)$ is continuous everywhere except at 0. d) $f(x) = \sin(\pi x)/\log|x^4|$ is continuous at $x = 0$. Is it continuous everywhere?

Example: The function $f(x) = [\sin(x + h) - \sin(x)]/h$ is continuous for every parameter $h > 0$. We will see soon what happens when h becomes smaller and smaller and that the continuity will never deteriorate but Indeed, we will see $f(x)$ will for smaller and smaller h get to the cos function.

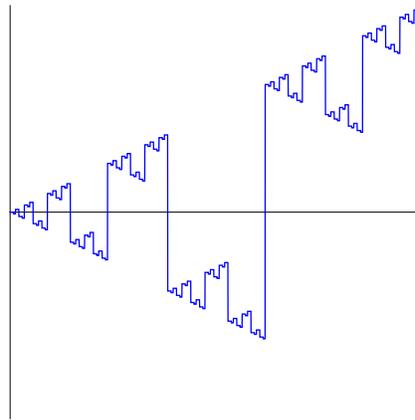
4.9. There are three major reasons, why a function can be not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.



4.10. Why do we like continuity? One reason important in applications is that “continuity provides stability and some sort of predictability.” Discontinuities are usually associated to catastrophes. Discontinuities happen typically, if something breaks.



This Weierstrass function is believed to be a fractal an object of dimension between 1 and 2. But it is continuous.



This function is discontinuous at every point. The vertical connection lines put for clarity are not part of the graph.

4.11. Continuity will be useful when finding maxima and minima. A continuous function on an interval $[a, b]$ has a maximum and minimum. We will see in the next hour that if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

4.12. Problem: Determine for each of the following functions, where discontinuities appear:

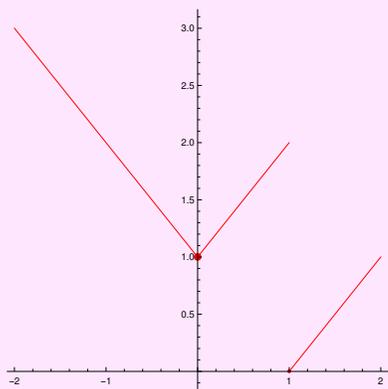
- a) $f(x) = \log(|x^2 - 1|)$
- b) $f(x) = \sin(\cos(\pi/x))$
- c) $f(x) = \cot(x) + \tan(x) + x^4$
- d) $f(x) = (x^2 + 2x + 1)/(x + 10 + (x - 1)^2/(x - 1))$
- e) $f(x) = \frac{x^2 - 4x}{x}$

Homework

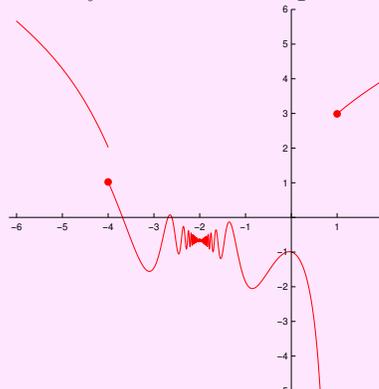
Problem 4.1: Which functions are continuous everywhere? Remember that we declare functions also continuous if one can fix “broken places” by assigning a value to an initially not defined place. Examples are $\sin(x)/x$ or $(x^2 - 1)/(x - 1)$ which are continuous everywhere when fixed.

- a) $\text{sinc}(5x) + \sin(x)/(2 + \sin^2(x)) + (x^3 - 1)/(x - 1)$, b) $\sin(\tan(x))$
- c) $\tan(\sin(x)) + \frac{x^2 + 5x + x^4}{x - 3}$
- d) $\text{sign}(x)/x$

Problem 4.2: On which intervals are the following functions continuous? You do not have to worry about end points.



a)



b)

Problem 4.3: Either do the following three problems a),b),c):

- a) Construct a function which has a jump discontinuity and an escape to infinity.
- b) Find a function which has an oscillatory discontinuity and an escape to infinity.
- c) Find a function which has a jump discontinuity as well as an oscillatory discontinuity.

or shoot down the problem with one strike:

Find a function which has a jump discontinuity, a pole and an oscillatory discontinuity all at the same time.

Problem 4.4: Heal the following functions to make them continuous everywhere or state that all hope is lost and that no healing is possible:

- a) $(x^3 - 27)/(x^2 - 6x + 9)$
- b) $\sin(x)(x^7 + x^5)/(x^2 + 3)$
- c) $((\sin(x))^3 - \sin(x))/(\cos(x) \sin(x))$.
- d) $(x^4 + 4x^3 + 6x^2 + 4x + 1)/(x^3 + 3x^2 + 3x + 1)$
- e) $(x^{70} - 1)/(x^{10} - 1)$

Problem 4.5: Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, experiment by plotting the functions.

- a) $\sin\left(\frac{1}{4+\sin(x)\cos(x)}\right) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 - \frac{13}{\exp(x)}$.
- b) $\frac{7}{\log|x|} + 5x^{77} - \cos(\sin(\cos(x))) - \exp(\log(\exp(x)))$