INTRODUCTION TO CALCULUS

MATH 1A

Unit 2: Functions

Lecture

2.1. A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^3 - 2x$ for example assigns to the number $x = 2$ the value $2^3 - 4 = 4$. A function is assigned a **domain** A, the points where f is defined and a **codomain** B a set of numbers in which f is mapped to. The **range** is $f(A)$.

2.2. Many functions like $f(x) = x^2 - 2x$ are defined everywhere. In general, we assume that the domain is the place where the function is defined and the codomain is the set of real numbers and the range the set of numbers which are reached by f. Functions can also be defined on domains which are discrete. The prime function $p(n)$ which gives the n'th prime is also a function. The domain is $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers, the range B is the set of primes.

2.3. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. Its range is also $\mathbb{R} \setminus \{0\}$, the set of real numbers without 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ for example is the inverse of the function $f(x) = x^2$ on its domain $\mathbb{R}^+ = [0, \infty)$. The function $f(x) = 1/x$ is its own inverse.

2.4. Here are a few examples. We will look at many of them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and the logarithmic function. Below we see some functions. The compound interest function can also be interpreted as an exponential. It will for $h \to 0$ go over to the exponential function. The logarithmic function as the inverse of the exponential function is only defined on the positive real axes.

constant	1	power	2^x
identity	x	exponential	$e^x = \exp(x)$
linear	$3x + 1$	logarithm	$\log(x) = \ln(x)$
quadratic	x^2	absolute value	$\lvert x \rvert$
cosine	$\cos(x)$	devil comb	$\sin(1/x)$
sine	$\sin(x)$	bell function	e^{-x^2}
compound interest	$\exp_h(x) = (1 + h)^{x/h}$	Agnesi	$\frac{1}{1+x^2}$
logarithms	$\log(x) = \ln(x)$	sinc	$\sin(x)/x$

We can build new functions by:

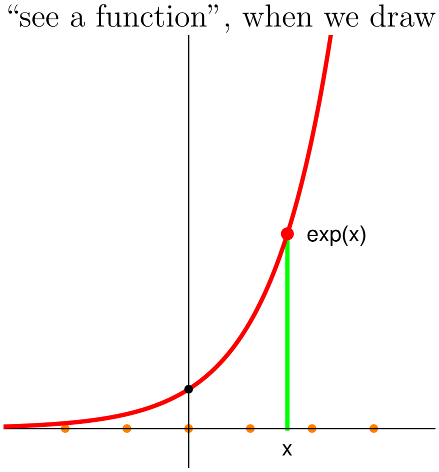
addition	$f(x) + g(x)$
multiplication	$f(x) * g(x)$
division	$f(x)/g(x)$
scaling	$2f(x)$
translation	$f(x+1)$
composing	$f(g(x))$
inverting	$f^{-1}(x)$

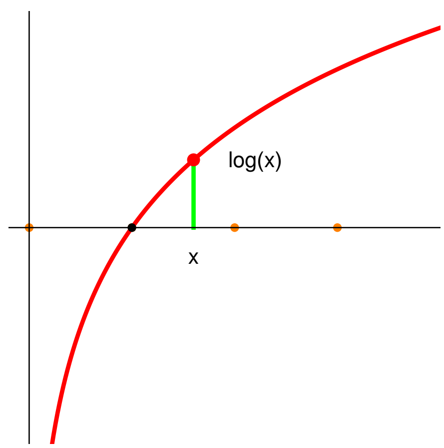
Important functions:

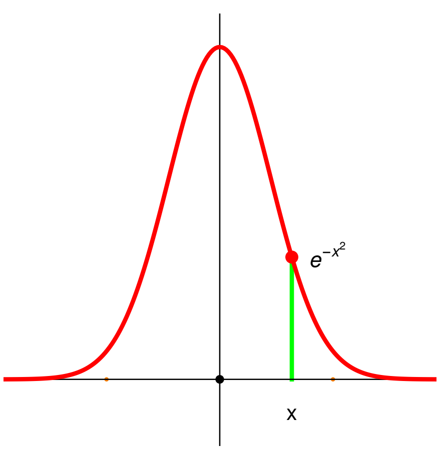
polynomials	$x^2 + 3x + 5$
rational functions	$(x+1)/(x^4+1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

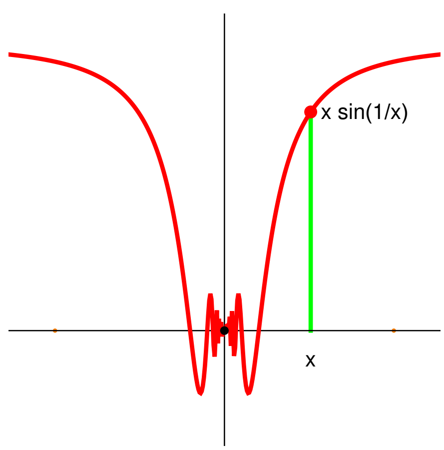
2.5. We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

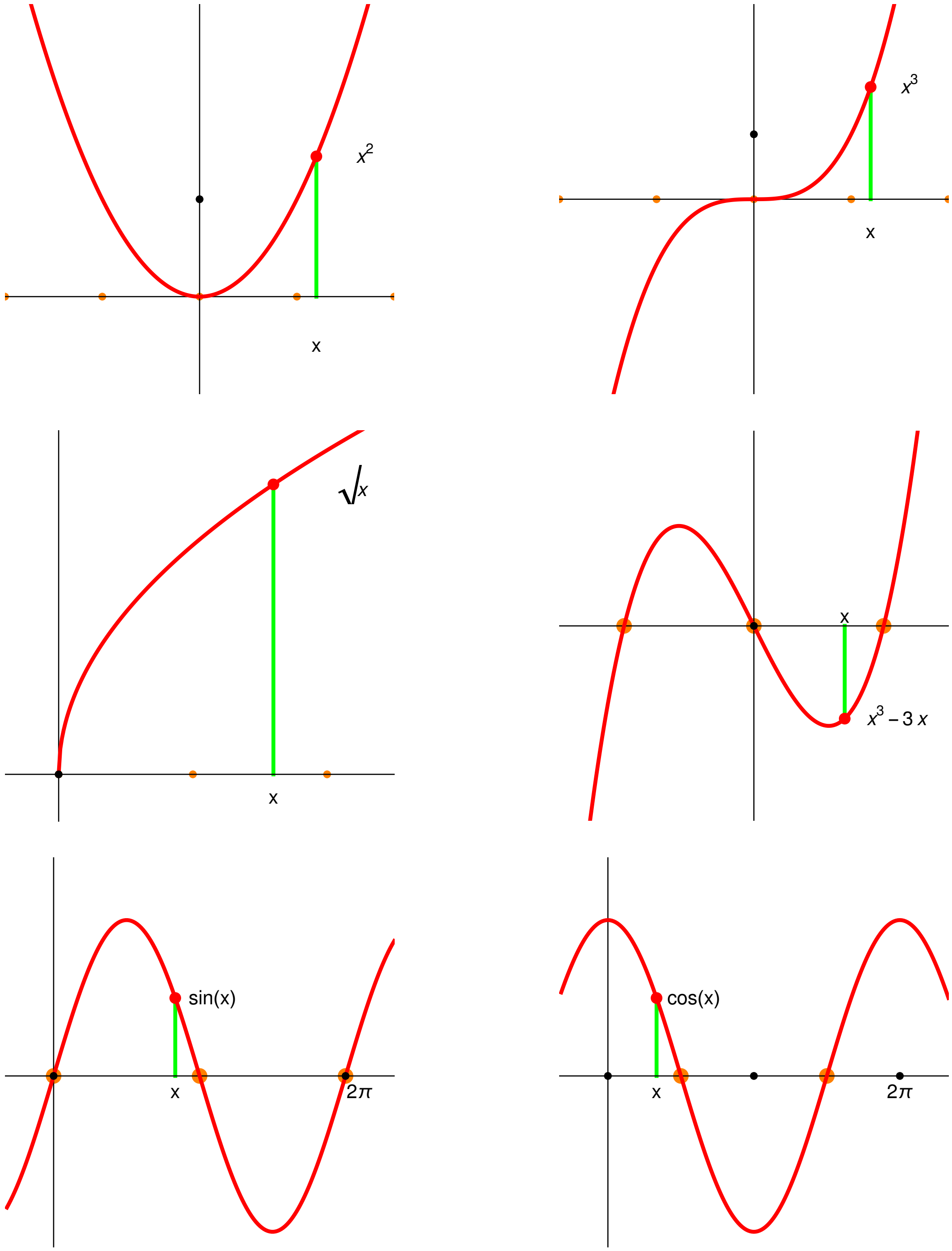
2.6. The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\ \}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see a function", when we draw the graph.











2.7. Definition: A function $f : A \to B$ is **invertible** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. The function g is the **inverse** of f. Example: $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$. You can check with the **horizontal line test** whether an inverse exists: draw the box with base A and side B, then every horizontal line should intersect the graph exactly once.

HOMEWORK

Here is the homework for this section.

Problem 2.1: Draw the function $f(x) = e^{x^2} \sin(4x)$ on the interval $[-5, 5]$. Its graph goes through the origin $(0, 0)$. You can use technology.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(-x) = f(x)$. Is f even?
c) What happens in general if a function f is both even and odd?

Problem 2.2: Determine from the following functions whether they are invertible. and write down the inverse if they are
a) $f(x) = x^{11} - 22$ from $A = \mathbb{R}$ to $B = \mathbb{R}$
b) $f(x) = \cos(x^5)$ from $A = [0, \pi/2]$ to $B = [0, 1]$
c) $f(x) = \sin(x)$ from $A = [0, \pi]$ to $B = [0, 1]$
d) $f(x) = \tan(x)$ from $A = (-\pi/2, \pi/2)$ to $B = \mathbb{R}$.
e) $f(x) = 1/(1 + x^2)$ from $A = [0, \infty)$ to $B = (0, 1]$.

Problem 2.3: a) Draw the graphs of $\exp_1(x) = 2^x, \exp_{1/4}(x) = (1 + 1/4)^{4x}$ and $\exp(x)$.
b) Draw the graphs the inverse of these functions.
You can use technology (like Desmos or Wolfram alpha) for a). For b), just "flip the graph" at the line $x = y$.

Problem 2.4: Try to plot the function $\exp(\exp(\exp(\exp(x))))$ on $[0, 1]$. You will see a sharp increase of the function after which the computer refuses to plot. Where is this value?

Problem 2.5: A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Find at least one root for each of the following functions.
a) $f(x) = x^7 - x^5$
b) $f(x) = \cos(x)$
c) $f(x) = 4\exp(-x^4)$
d) $f(x) = \log(x) = \ln(x)$
e) $f(x) = \sin(x) - 1$
f) $f(x) = \sec(x) = 1/\cos(x)$

(*) Here is how you to plot a function with Wolfram alpha:

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http://www.wolframalpha.com/input/?i=Plot+sin(x)
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