Data project 3: Chaos

3.1. Given a function, like $f(x) = 4x(1 - x)$, we can look for points where $f(x) = x$. This is called a **fixed point**. You can see that in our case, $x = 0$ and $x = 3/4$ are the two solutions. The function $f(f(x))$ is a new function and in our example again a polynomial. Now, if $f(f(x)) = x$, then we have a periodic point of period 2 because if we apply f to such a point, we return after 2 steps, $x, f(x), f(f(x)) = x$.

3.2. Problem 1: when expanding $f(f(x))$ we get a polynomial of degree 4. $f(f(x)) = 4(4x(1 - x))(1 - (4x(1 - x))) = 16x - 80x^2 + 128x^3 - 64x^4$. Find all the roots of this function. You can use that you already know two of them since if $f(x) = x$, then also $f(f(x)) = x$.

3.3. Let us start with $x = 0.4$ and apply $f(x) = 4x(1 - x)$. We get 0.96. Now compute $f(0.96) = 1.536$. We do that 60 times. Here is the Mathematica code which does that.

```
f[x_] := 4x(1−x); NestList[f, 0.4, 60]
```

Running this code produces the following data 0.4, 0.96, 0.1536, 0.520028, 0.998395, 0.00640774, 0.0254667, 0.0992726, 0.35767, 0.918969, 0.29786, 0.836557, 0.546917, 0 0.991195, 0.034909, 0.134761, 0.466403, 0.995485, 0.0179785, 0.0706211, 0.262535, 0.774441, 0.698727, 0.84203, 0.532063, 0.995888, 0.0163812, 0.0644512, 0.241189, 0.732068, 0.784578, 0.676061, 0.87601, 0.434465, 0.982821, 0.067536, 0.251899, 0.753784, 0.742374, 0.76502, 0.719059, 0.808053, 0.620412, 0.942003, 0.218532, 0.683103, 0.865893, 0.464488, 0.994956, 0.0200758, 0.0786912, 0.289996, 0.823593, 0.581151, 0.973658, 0.102592, 0.368269, 0.930587, 0.258378, 0.766475, 0.715963.

3.4. We start again with $x = 0.4$ again but apply the function $g(x) = 4x - 4x^2$. We get 0.96. Now compute $f(0.96) = 1.536$ etc. We do that 60 times. Here is the Mathematica code which does that.

```
g[x_] := 4x−4x^2; NestList[g, 0.4, 60]
```

Running this line of code produces the following data 0.4, 0.96, 0.1536, 0.520028, 0.998395, 0.00640774, 0.0254667, 0.0992726, 0.35767, 0.918969, 0.29786, 0.836557, 0.546917, 0.991195, 0.034909, 0.134761, 0.466403, 0.995485, 0.0179785, 0.0706211, 0.262535, 0.774441, 0.698727, 0.84203, 0.532063, 0.995888, 0.0163812, 0.0644512, 0.241189, 0.732068, 0.784578, 0.676061, 0.87601, 0.434465, 0.982821, 0.0675375, 0.251905, 0.753795, 0.742353, 0.76506, 0.71 8973, 0.808203, 0.620042, 0.942359, 0.217273, 0.680263, 0.870022,

0.452336, 0.990913, 0.0360196, 0.138889, 0.478395, 0.998133, 0.00745456, 0.029596, 0.11488, 0.406731, 0.965204, 0.134342, 0.465178, 0.99515.

3.5. Problem 2) a) What is algebraically the difference between $f(x)$ and $g(x)$? Are you surprised when you look at the data? We have prepared a Desmos applet

https://www.desmos.com/calculator/xvh5ae0a1a

For you to explore that. b) What does Desmos give for $f^{(60)}(0.4)$? c) What result does Desmos give for $g^{(60)}(0.4)$? Note that we write $f^{(2)}(x) = f(f(x))$ and $f^{(3)}(x) = f(f(f(x)))$ etc.

3.6. As the title of this project shows, the reason for this strange behavior is "chaos". Chaos is defined as "sensitive dependence on initial conditions".

3.7. We now take a graphing calculator or the scientific calculator on your phone to experiment with iterating functions.

Problem 3) a) What happens if you push the $\boxed{\cos}$ button repetitively. b) What happens if you push alternating the $\boxed{\tan}$ and $1/x$ button? (This actually means that we are iterating the cot function but there is no cot button in general on calculators).
Problem 4) a) Who was Mitchell Feigenbaum? b) What did he discover? c) Where is his discovery applied?

Problem 5) a) What is the **Feigenbaum bifurcation diagram** for the logistic map $f_c(x) = cx(1 - x)$ describe? b) What is the **Feigenbaum constant?**
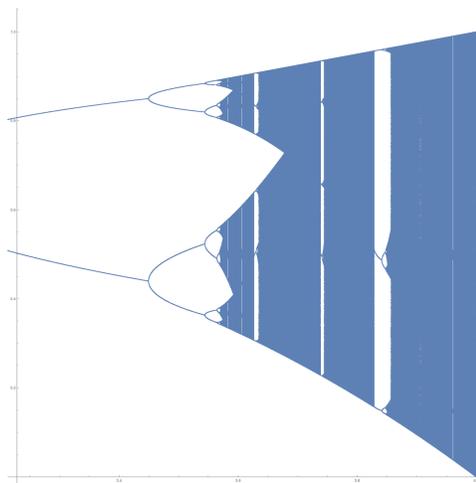

FIGURE 1. Feigenbaum bifurcation diagram