Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice E

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The quantum exponential function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$ for $h > 0$.

Solution:
This is an extremely important identity because it leads to the property $\exp' = \exp$.

2) T F The function $\text{sinc}(x) = \sin(x)/x$ has a critical point at $x = 0$.

Solution:
Take the derivative $[x\cos(x) - \sin(x)]/x^2$ and apply l'Hospital to get 0. This was not an easy TF problem.

3) T F The limit of $1/\log(1/|x|)$ for $x \to 0$ exists.

Solution:
We have seen this in class and in a midterm. Since $\log(1/|x|) = -log|x|$ goes to infinity for $|x| \to 0$, we know that $1/\log(1/|x|)$ converges to 0.

4) T F The strawberry theorem tells that for any $f(x)$, its anti-derivative $F(x)$ and $g(x) = F(x)/x$ the points $f = g$ are the points where $g' = 0$.

Solution:
Yes, that is the theorem. Tasty!

5) T F The function $f(x) = \tan(x)$ has a vertical asymptote at $x = \pi/2$.

Solution:
Yes, if the angle x goes to $\pi/2$, then this means the slope goes to infinity.

6) T F The function $x/(1+x)$ converges to 1 for $x \to \infty$ and has therefore a horizontal asymptote.

Solution:
We could use the Hospital rule to see that the limit $x \to \infty$ is $1/1 = 1$. It can also be seen intuitively. If x=1000 for example, we have 1000/1001.

7) T F The function $f(x) = \tan(x)$ is odd: it satisfies $f(x) = -f(-x)$.

Solution:
Yes, it appears odd but it is true: tan is odd. Even tomorrow.

8) T F The function $\sin^3(x)/x^2$ is continuous on the real line.

Solution:
It can be written as $\text{sinc}(x)^2 \sin(x)$, a product of two continuous functions.

9) T F With $Df(x) = f(x+1) - f(x)$ we have $D(fg)(x) = Dfg(x+1) + f(x)Dg(x)$.

Solution:
This is also called the quantum Leibniz rule.

10) T F If f has a critical point 0 then f has a minimum or maximum at 0.

Solution:
The function $f(x) = x^3$ is a counter example.

11) T F The limit of $[\frac{1}{3+h} - \frac{1}{3}]/h$ for $h \to 0$ is $-1/9$.

Solution:
This is the derivative of $1/x$ at $x = 3$.

12) T F — The function $(\cos(x)+\sin(3x))/(\sin(4x)+\cos(3x))$ can be integrated using trig substitution.

Solution:
Yes, this works with the magic substitution $u = \tan(x/2)$, $dx = \frac{2du}{(1+u^2)}$, $\sin(x) = \frac{2u}{1+u^2}$ and $\cos(x) = \frac{1-u^2}{1+u^2}$.

13) T F — The marginal cost is the anti-derivative of the total cost.

Solution:
It is the derivative of the total cost, not the anti-derivative.

14) T F — The cumulative distribution function is the anti-derivative of the probability density function.

Solution:
Yes, now we are talking.

15) T F — The function $\sqrt{1-x^2}$ can be integrated by a trig substitution $x = \cos(u)$.

Solution:
For example, one could also take $x = \sin(u)$.

16) T F — The integral $\int_0^1 1/x^2 \, dx$ is finite.

Solution:
The anti-derivative of the function inside the integral is $1/x$ which does not look good in the limit $x \to 0$. Nope, the integral does not exist. Intuitively, $1/x^2$ just goes to infinity too fast if $x \to 0$.

17) T F — The chain rule tells that $d/dx f(g(x)) = f'(x)g'(x)$.

Solution:
There is a missing link in the chain. Look it up.

18) | T | F | For the function $f(x) = \sin(100x)$ the hull function is constant.

Answer marked: T

Solution:
Yes the maxima are at 1. Connecting the maxima gives the line $x = 1$. The lower hull is $x = -1$. —

19) | T | F | The trapezoid method is also called Simpson rule.

Answer marked: F

Solution:
No, the Simpson rules are more sophisticated and use 2 or 3 values in between the interval.

20) | T | F | If $f''(x) > 0$, then the curvature of f is positive.

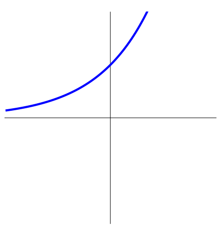
Answer marked: T

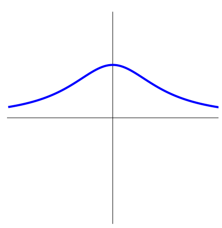
Solution:
Indeed, the curvature is defined as $\frac{f''(x)}{(1+f'(x)^2)^{3/2}}$.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	


a)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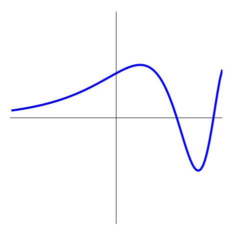
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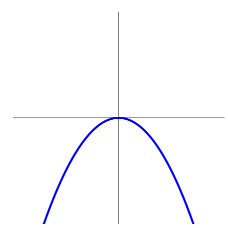
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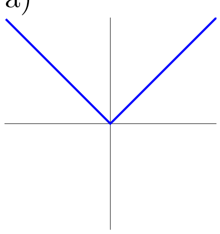
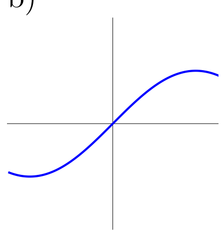

e)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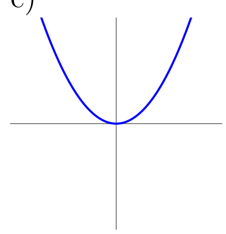
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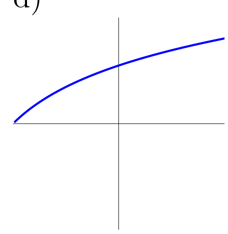
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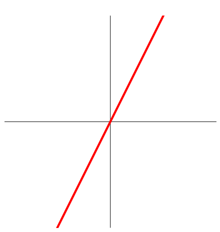
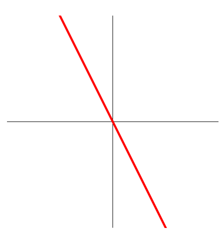

1)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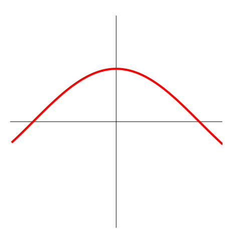
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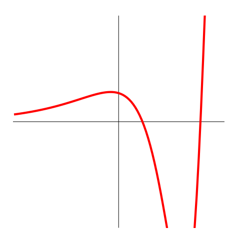
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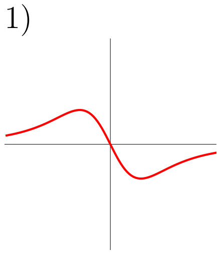
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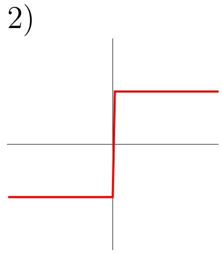
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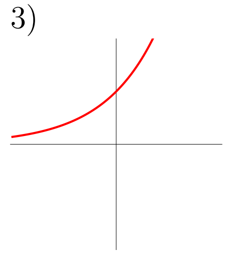
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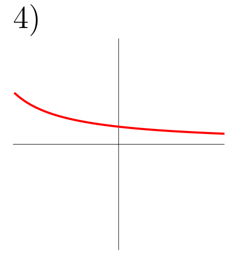
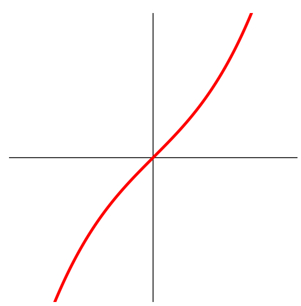

8)

Solution:

Function	Fill in the numbers 1-8
graph a)	7
graph b)	5
graph c)	4
graph d)	2
graph e)	6
graph f)	3
graph g)	1
graph h)	8

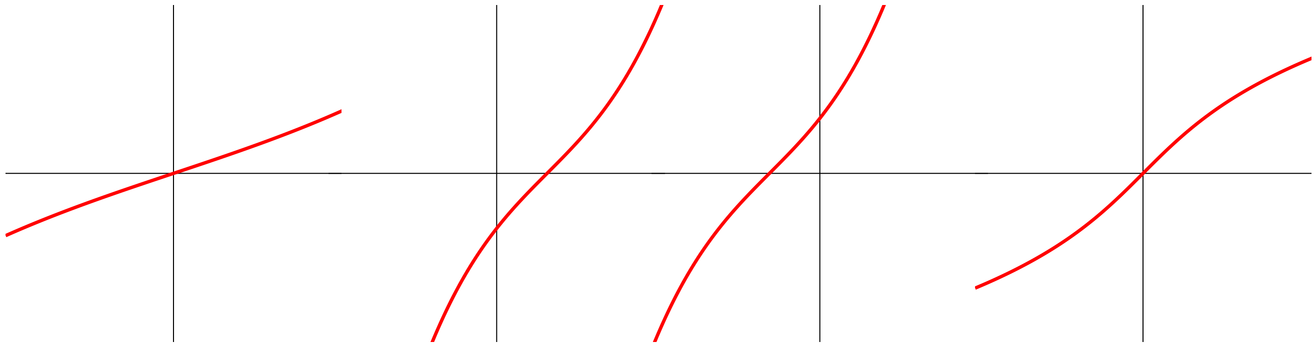
Problem 3) Matching problem (10 points) No justifications are needed.

Here is the graph of a function $f(x)$. Match the following modifications

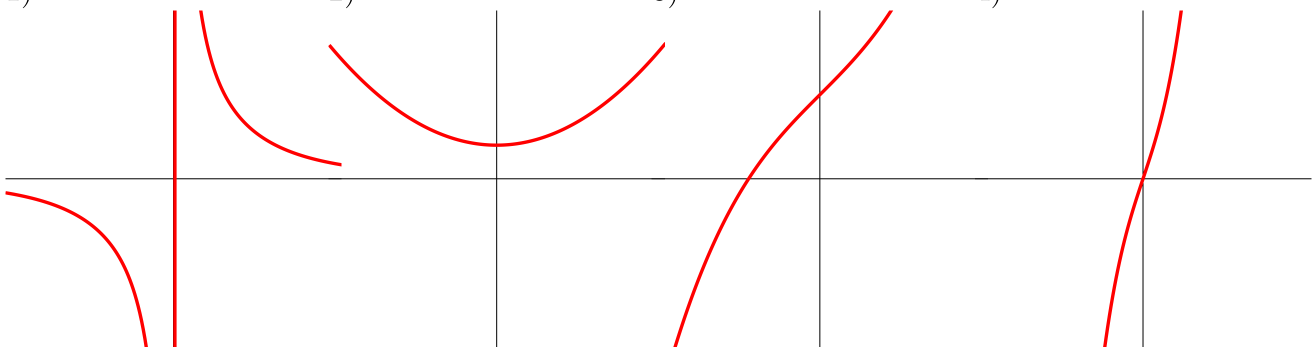


Match the following functions with their graphs.

Function	Fill in 1)-8)
$f(x-1)$	
$f'(x)$	
$f(x+1)$	
$f^{-1}(x)$	
$f(x/2)$	
$f(3x)$	
$1/f(x)$	
$f(x)+1$	



1) 2) 3) 4)

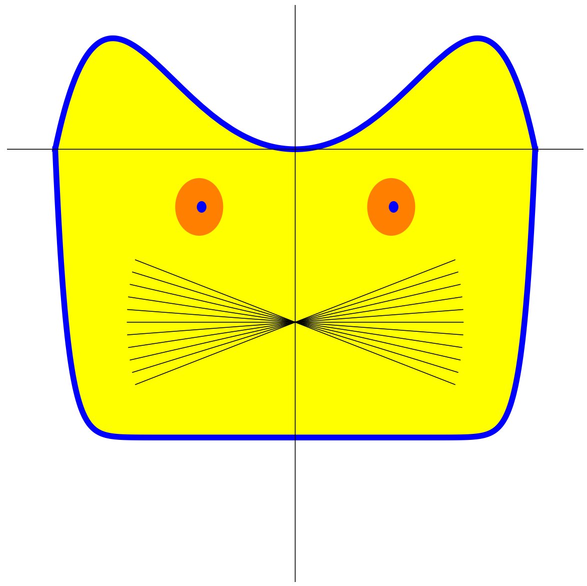


5) 6) 7) 8)

Solution:
2,6,3,4,1,8,5,7

Problem 4) Area computation (10 points)

Find the area of the **cat region** which is the region enclosed by the functions $f(x) = x^{20} - 1$ and $g(x) = x^2 - x^6$. No need to count in the whiskers.

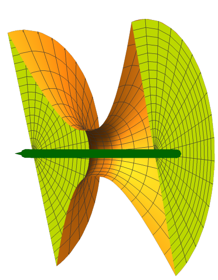


Solution:
We first have to find where the two graphs intersect to determine the integration bounds. They intersect at $x = 1$ and $x = -1$. Next, we have to know which function is above and which is below. If we look at $x = 0$, then we see that f is below. We can see this also from the fact that f is always nonpositive and g is always non-negative. Therefore,

$$\int_{-1}^{1} x^2 - x^6 - (x^{20} - 1)\, dx = (x + x^3/3 - x^7/7 - x^{21}/21)|_{-1}^{1} = 16/7 .$$

Problem 5) Volume computation (10 points)

We spin the graph of the function $f(x) = \sqrt{1 + |x|^3}$ around the x axes and get a solid of revolution. What is the volume of this solid enclosed between $x = -3$ and $x = 3$? The picture shows half of this sold.



Solution:
The area at position x is $\pi f(x)^2 = \pi(1 + |x|^3)$. The volume of a slice of thickness dx is $\pi(1 + |x|^3)\ dx$. We have to integrate this from $x = -3$ to $x = 3$. To avoid the absolute value, we take twice the integral from 0 to 3 and have

$$2\pi \int_0^3 (1 + x^3)\ dx = 2\pi(x + x^4/4)|_0^3 = 2\pi 93/4 = \pi 93/2\ .$$

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals

a) $\int_{-1}^{1} \frac{1}{1+x^2}\ dx$

b) $\int_1^2 x^2 + \sqrt{x}\ dx$

c) $\int_0^{\sqrt{\pi}} \sin(x^2) 2x\ dx$.

d) $\int_0^1 \log(4 + x)\ dx$.

Solution:
a) The anti-derivative is $\arctan(x)$. Evaluated at 1 it is $\pi/4$ at -1 it is $-\pi/4$. The integral is $\pi/2$.
b) The integral is $(x^3/3 + x^{3/2}(2/3))|_1^2 = (5 + 4\sqrt{2})/3$.
c) $-\cos(x^2)|_0^{\sqrt{\pi}} = 2$.
d) Write $4 + x = u$ and get $\int_4^5 \log(u)\ du = u \log(u) - u|_4^5 = 5 \log(5) - 4 \log(4) - 1$.

Problem 7) Extrema (10 points)

a) (7 points) Analyse the local extrema of the function

$$f(x) = \frac{x}{1 + x^2}$$

on the real axes using the second derivative test.

b) (3 points) Are there any global extrema?

Solution:
a) The derivative is

$$f'(x) = \frac{1-x^2}{1+x^2)^2} .$$

The extrema are $x = 1$ and $x = -1$. The second derivative is

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3} .$$

b) Asymptotically, we have $f(x) \to 0$ for $|x| \to \infty$. This means that $x = 1$ is a global maximum and $x = -1$ is a global minimum.

Problem 8) Integration by parts (10 points)

a) (5 points) Find the anti-derivative of

$$f(x) = \sin(4x)\cos(3x) .$$

b) (5 points) Find the anti-derivative of

$$f(x) = (x-1)^2 \sin(1+x) .$$

Solution:
a) This is a problem, where we have to do integration by parts twice. It is a "merry go round problem". Call I the integral. Then do integration by parts twice to isolate $I = \int \sin(4x)\cos(3x)\,dx$. We will also encounter $J = \int \cos(4x)\sin(3x)\,dx$. Then integration by parts gives

$$I = \sin(4x)\sin(3x)/3 - (4/3)J$$

$$J = -\cos(4x)\cos(3x)/3 - (4/3)I$$

so that $I = \sin(4)\sin(3x)/3 + (4/9)\cos(4x)\cos(3x)) + (16/9)I$ so that $I(1-16/9) = \sin(4)\sin(3x)(1/3) + (4/9)(\cos(4x)\cos(3x))$ and

$$I = [\sin(4x)\sin(3x)(1/3) + (4/9)(\cos(4x)\cos(3x))](-9/7) .$$

b) Use integration by parts twice. We end up with

$$-(x-1)^2\cos(1+x) + (2x-2)\sin(1+x) + 2\cos(1+x) .$$

Problem 9) Substitution (10 points)

a) (3 points) Find the integral $\int 3x\sqrt{5x^2-5}\,dx$.

b) (3 points) What is the anti-derivative of $\int \exp(x^2-x)(4x-2)$?

c) (4 points) Evaluate the definite integral

$$\int_0^{\pi/2} \sqrt{1-\cos(x)}\sin(x)\;dx\;.$$

Solution:
a) $(5x^2-5)^{3/2}/5+C$.
b) $2\exp(x^2-x)+C$.
c) $(1-\cos(x))^{3/2}(2/3)|_0^{\pi/2}=2/3$.

Problem 10) Partial fractions, Trig substitution (10 points)

a) Solve the integral

$$\int \frac{2-x+x^2}{(1-x)(1+x^2)}$$

by writing

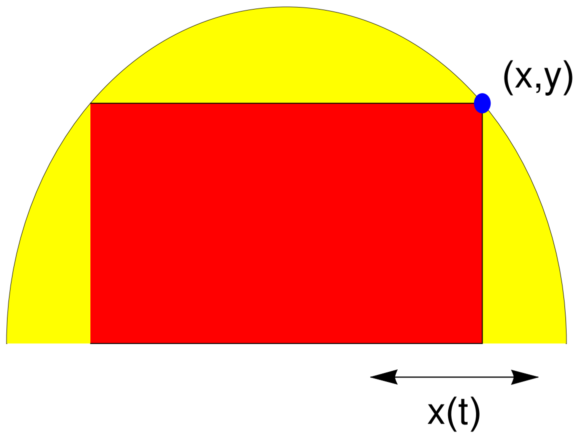
$$\frac{2-x+x^2}{(1-x)(1+x^2)}=\frac{A}{1+x^2}+\frac{B}{1-x}\;.$$

b) Evaluate the integral $\int \sqrt{1-x^2}x\;dx$.

Solution:
a) $\arctan(x)-\log(x-1)$.
b) Write $x=\sin(u)$ to get $\int \cos^2(u)\sin(u)\;du=-\cos(u)^{3/2}/3$. This can also be solved more easily by substitution: $u=1-x^2, du=-2xdx$ etc.

Problem 11) Differentiation rules (10 points)

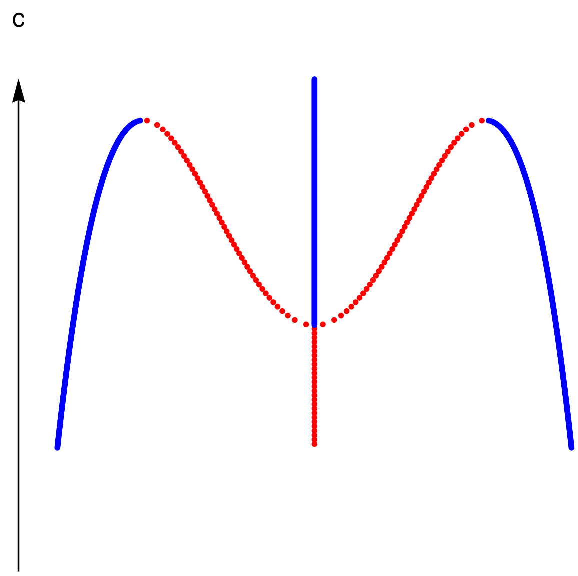
A rectangle with corners at $(-x,0),(x,0),(x,y),(-x,y)$ is inscribed in a half circle $x^2+y^2=1$ where $y\geq 0$ is in the upper half plane. We know $x(t)=\cos(t)$ and $y(t)=\sin(t)$. Find the rate of change of the area $2x(t)y(t)$ of the rectangle.

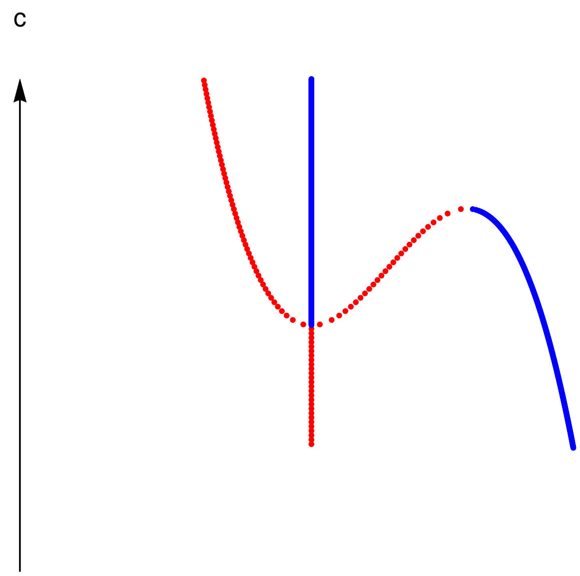


Solution:
$A(t) = 2\cos(t)\sin(t) = 2\sin(2t)$. The derivative is $4\cos(2t)$.

Problem 12) Catastrophes (10 points)

The following two pictures show bifurcation diagrams. The vertical axes is the deformation parameter c. On the left hand side, we see the bifurcation diagram of the function $f(x) = x^6 - x^4 + cx^2$, on the right hand side the bifurcation diagram of the function $f(x) = x^5 - x^4 + cx^2$. As done in class and homework, the bolder continuously drawn graphs show the motion of the local minima and the lighter dotted lines show the motion of the local maxima. In both cases, determine the catastrophe for the criticaal point $x = 0$.





Solution:
Bifurcation parameters are parameter values where a local minimum disappears. In the first picture, the bifurcations happen at $c = 0$.

Problem 13) Applications (10 points)

The **Laplace distribution** is a distribution on the entire real line which has the probability density $f(x) = e^{-|x|}/2$. The variance of this distribution is the integral

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx \ .$$

Find it.

Solution:
The integral is $\int_0^\infty x^2 e^{-x}$. We compute this using integration by parts:

x^2	$\exp(-x)$	
$2x$	$-\exp(-x)$	$\oplus$
2	$\exp(-x)$	$\ominus$
0	$-\exp(-x)$	$\oplus$

The integral is $2(-x^2-2x-2)e^{-x}|_0^\infty$. The definite integral $2\int_0^\infty$ is 4. The original function had $e^{-x}/2$. The final result is $\boxed{2}$.