Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice D

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F (answer: F)

The definite integral $\int_0^{2\pi} \sin^2(5x)\ dx$ is zero.

Solution:
The integrand is never negative and almost everywhere positive.

2) T F (answer: F)

The intermediate value theorem assures that the function $\exp(\sin(x))$ has a root in the interval $(0, 2\pi)$.

Solution:
The function $\exp(\sin(x))$ is never zero.

3) T F (answer: T)

$\frac{d}{dx} \cos(4x) = -4\sin(4x)$.

Solution:
differentiate

4) T F (answer: F)

If $f''(1) < 0$ then 1 is a local maximum of f.

Solution:
It also has to be a critical point.

5) T F (answer: F)

The derivative of $1/x$ is $\log(x)$ for all $x > 0$.

Solution:
It is the anti-derivative, not the anti derivative

6) T F (answer: T)

The limit of $\sin(3x)/(5x)$ for $x \to 0$ exists and is equal to $3/5$.

Solution:
Use Hôpital

7) T F The function $(e^t - 1)/t$ has the limit 1 as t goes to zero.

Solution:
Use Hopital

8) T F The derivative of $f(f(x))$ is $f'(f'(x))$ for any differentiable function f.

Solution:
This is not the chain rule

9) T F A monotonically increasing function f has no point x, where $f'(x) < 0$.

Solution:
Increasing means that the derivative is positive.

10) T F The function $f(x) = \exp(-x^2)$ has an inflection point x somewhere on the real line.

Solution:
The second derivative can be zero. One can see this by looking at the graph.

11) T F The function $f(x) = (1 - x^3)/(1 + x)$ has a limit for $x \to -1$.

Solution:
The top $1 - x^3$ is not zero at $x = -1$ so that the function has a pole

12) T F If we know the marginal cost for all quantities x as well as the total cost for $x = 1$ we know the total cost for all x.

Solution:
We can form the anti derivative and fix the constant from $F(1)$.

13) T F

The function f which satisfies $f(x) = 0$ for $x < 0$ and $f(x) = e^{-x}$ for $x \geq 0$ is a probability density function.

Solution:
True, it is nonnegative every where and the total integral is 1.

14) T F

The differentiation rule $(f \cdot g)' = f'(g(x)) \cdot g'(x)$ holds for all differentiable functions f, g.

Solution:
We would need the Leibniz product rule, not the chain rule.

15) T F

Hôpital's rule assures that $\cos(x)/\sin(x)$ has a limit as $x \to 0$.

Solution:
The nominator does not go to zero for $x \to 0$.

16) T F

A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

Solution:
By definition

17) T F

The family of functions $f_c(x) = cx^2$ where c is a parameter has a catastrophe at $x = 0$.

Solution:
For $c < 0$ we have a local max, for $c > 0$ we have a local min.

18) T F

The fundamental theorem of calculus implies $\int_{-x}^{x} f'(t)\, dt = f(x) - f(-x)$ for all differentiable functions f.

Solution:
Yes, this is the most important result in this course.

19) T F

If f is a smooth function for which $f''(x) = 0$ everywhere, then f is constant.

Solution:
It can be linear

20) T F

The function $f(x) = \sin(x)/(1 - \cos(x))$ can be assigned a value $f(0)$ such that $f(x)$ is continuous at 0.

Solution:
Use l'Hopital to see that the limit is the same as $\lim_{x \to 0} \cos(x)/\sin(x)$ which has no limit at $x = 0$.

Problem 2) Matching problem (10 points) Only short answers are needed.

We name some important concepts in this course. To do so, please complete the sentences with one or two words. Each question is one point.

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is called the		of f.
$f'(x) = 0, f''(x) > 0$ implies that x is a		of f.
The sum $\frac{1}{n}[f(0) + f(1/n) + f(2/n) + \cdots + f((n-1)/n) + f(1)]$ is called a		sum.
If $f(0) = -3$ and $f(4) = 8$, then f has a root on the interval $(0, 4)$ by the		theorem.
There is a point $x \in (0, 1)$ where $f'(x) = f(1) - f(0)$ by the		theorem.
The expansion rate $r'(t)$ can be obtained from $d/dtV(r(t)) = -5$ by the differentiatl rule called		rates.
The anti derivative $\int_{-\infty}^{x} f(t)\ dt$ of a probability density function f is called the		function.
A point x for which $f(x) = 0$ is called a		of f.
A point x for which $f''(x) = 0$ is called an		of f.
At a point x for which $f''(x) > 0$, the function is called		up.

Solution:
Derivative
Local minimum
Riemann sum
Intermediate value
Mean value
Chain rule
Cumulative distribution
Root
Inflection Point
Concave

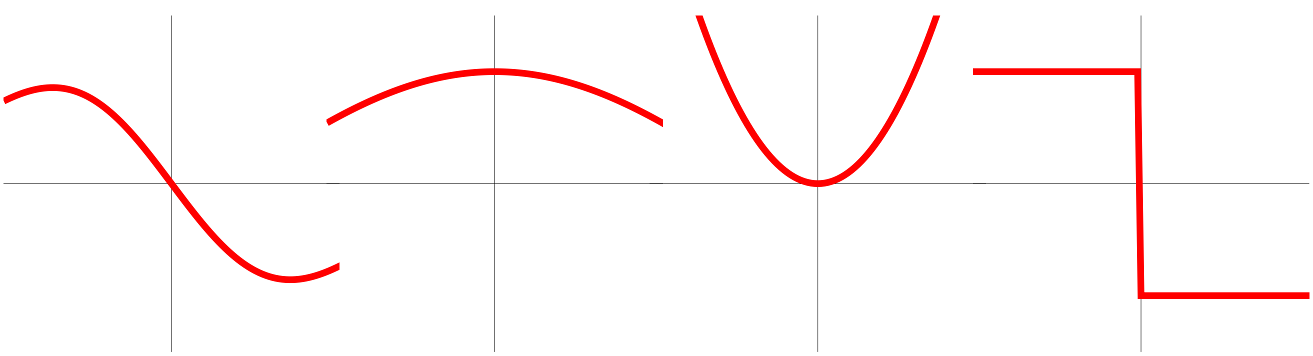
Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) Find the relation between the following functions:

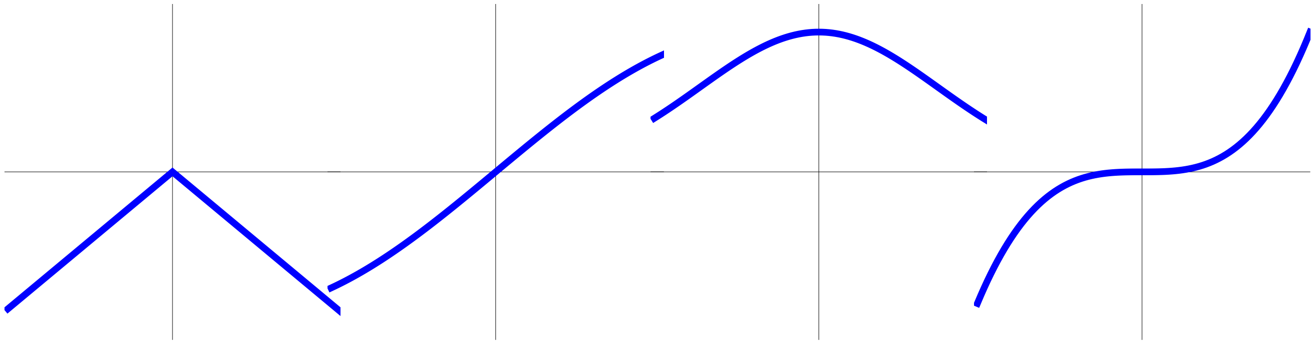
function f	function g	$f = g'$	$g = f'$	none
$\log\|\sin(x)\|$	$\cot(x)$			
$1/\cos^2(x)$	$\tan(x))$			
x^5	$5x^4$			
$1/x^2$	$-1/x$			
$\sin(\log(x))$	$\cos(\log(x))/x$			

b) (3 points) Match the following functions (a-d) with a choice of **anti-derivatives** (1-4).

Function a)-d)	Fill in 1)-4)
graph a)	
graph b)	
graph c)	
graph d)	



a) b) c) d)



1) 2) 3) 4)

c) (3 points) Find the limits for $x \to 0$

Function f	$\lim_{x\to 0} f(x)$
$x/(e^{2x}-1)$	
$(e^{2x}-1)/(e^{3x}-1)$	
$\sin(3x)/\sin(5x)$	

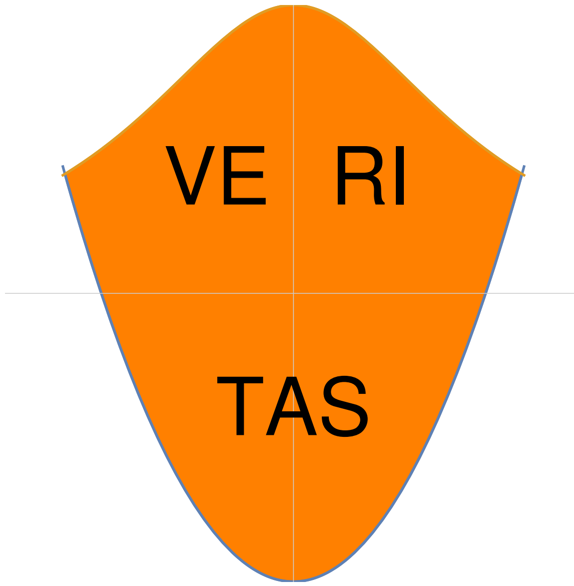
Solution:
a)

function f	function g	$f = g'$	$g = f'$	none
$\log\|\sin(x)\|$	$\cot(x)$		*	
$1/\cos^2(x)$	$\tan(x))$	*		
x^5	$5x^4$		*	
$1/x^2$	$-1/x$	*		
$\sin(\log(x))$	$\cos(\log(x))/x$		*	

b) 3,2,4,1
c) Use l'Hopital: 1/2,2/3,3/5

Problem 4) Area computation (10 points)

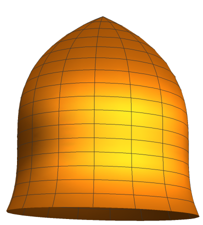
Find the area of the shield shaped region bound by the two curves $1/(1+x^2)$ and x^2-1.



Solution:
The two curves intersect at $x = \pm 2^{1/4}$.
$\int_{-2^{1/4}}^{2^{1/4}} \frac{1}{1+x^2} - x^2 + 1 \, dx = \arctan(x) - x^3/3 + x|_{-2^{1/4}}^{2^{1/4}} = 2\arctan(2^{1/4}) - (2/3)2^{3/4} + 2 \cdot 2^{1/4}$.

Problem 5) Volume computation (10 points)

Did you know that there is a scaled copy of the **liberty bell** on the campus of the Harvard business school? Here we compute its volume. Find the volume of the rotationally symmetric solid if the radius $r(z)$ at height z is $r(z) = 8 - (z-1)^3$ and the height z of the bell is between 0 and 3.



Solution:

$$\pi \int_0^3 \pi(8-(z-1)^3)^2 \, dz = \pi \int_{-1}^2 (8-u^3)^2 \, du = \pi \int_{-1}^2 64-16u^3+u^6 \, du = \pi(64z-16u^4/4+u^7/7)|_{-1}^3 = \pi 1053/7$$

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{1}{x^4}\, dx \; .$$

b) (5 points) Find the integral or state that it does not exist

$$\int_1^\infty \frac{1}{x^{3/2}}\, dx \; .$$

Solution:
a)

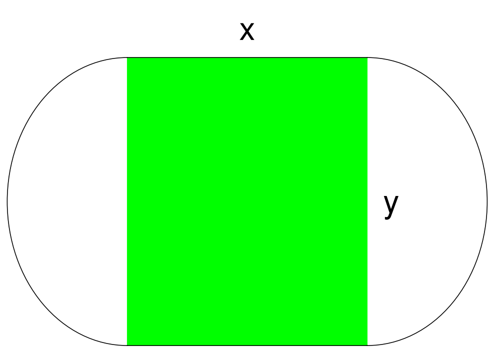
$$\int_1^\infty \frac{1}{x^4}\, dx = \frac{-x^{-3}}{3}\Big|_1^\infty = \frac{1}{3} \; .$$

b)

$$\int_1^\infty \frac{1}{x^{3/2}}\, dx = -2x^{-1/2}|1^\infty = 2 \; .$$

Problem 7) Extrema (10 points)

The Harvard stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?



Solution:
Solve for $y = (200 - x)/\pi$ and plug this into the function to get

$$f(x) = xy = x(200 - x)/\pi \ .$$

To find the maximum of this function, we differentiate with respect to x and look where the derivative is zero:

$$f'(x) = (200 - 2x)/\pi = 0$$

showing that $x = 100$ is the maximum.

Problem 8) Integration by parts (10 points)

Find the antiderivative:

$$\int (x-1)^4 \exp(x+1) \, dx \ .$$

Solution:
Use the Tic-Tac-Toe integration method:

$(x-1)^4$	$\exp(x+1)$	
$4(x-1)^3$	$\exp(x+1)$	$\oplus$
$12(x-1)^2$	$\exp(x+1)$	$\ominus$
$24(x-1)$	$\exp(x+1)$	$\oplus$
24	$\exp(x+1)$	$\ominus$
0	$\exp(x+1)$	$\oplus$

Adding things up gives

$$e^{x+1}[(x-1)^4 - 4(x-1)^3 + 12(x-1)^2 - 24(x-1) + 24] \ .$$

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int e^{x^2} 2x \, dx$.

b) (3 points) Solve the integral $\int 2x \log(x^2) \, dx$.

c) (4 points) Find the integral $\int e^{-2e^x} e^x \, dx$.

Solution:
These are all standard substitution problems:
a) $e^{x^2} + c$
b) $x^2 \log(x^2) - x^2 + c$
c) $-e^{-2x^2}/2 + c$

Problem 10) Partial fractions (10 points)

a) (5 points) Find the definite integral

$$\int_1^5 \frac{1}{(x-4)(x-2)} \, dx \; .$$

b) (5 points) Find the indefinite integral

$$\int \frac{1}{(x-1)(x-3)(x-5)} \, dx \; .$$

Solution:
In both problems we can find the coefficients quickly with the l'Hopital method: a) $\int_1^5 \frac{1}{(x-4)(x-2)} \, dx = \frac{1}{2} \int_1^5 [\frac{1}{x-4} - \frac{1}{x-2}] \, dx = \frac{1}{2}[\log|x-4| - \log|x-2|]|_1^5 = -\log(3)$. b) The factorization

$$\frac{1}{(x-1)(x-3)(x-5)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-5}$$

can be obtained quickly from l'Hopital: $A = \lim_{x\to 1} \frac{1}{(x-3)(x-5)} = \frac{1}{8}$ and $B = \lim_{x\to 3} \frac{1}{(x-1)(x-5)} = -\frac{1}{4}$ and $C = \lim_{x\to 5} \frac{1}{(x-1)(x-3)} = \frac{1}{8}$ so that the result is

$$[\log|x-1| - 2\log|x-3| + \log|x-5|]/8 \; .$$

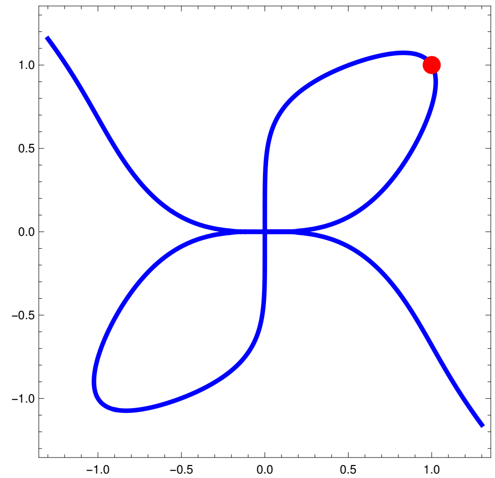
[P.S. As in the homeworks, we do not worry in a) that these are improper integrals, integrating over the logarithmic singularity. They are no problem because the integral of $\log|x|$ is $x\log|x| - x$ which has a limit 0 for $x \to 0$.]

Problem 11) Chain rule (10 points)

The coordinates of a car on a freeway intersection are $x(t)$ and $y(t)$. Use the chain rule to differentiate

$$f(t) = x(t)^7 + y(t)^7 - 2x(t)y(t)^2 \; .$$

Assume we know $x'(0) = 3$ and $x(0) = 1, y(0) = 1$. Find the derivative $y'(t)$.



Solution:
Differentiate the relation with respect to t and solve for y':

$$7x^6x' + 6y^2y' - 2x'y^2 - 4xyy' = 0 \; .$$

Therefore, we can solve for

$$y' = (7x^6x' - 7y^6x')/(4xy - 6y^2) \; .$$

The final answer is $\boxed{-5}$.

Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (2 points) $f(x) = \sin^5(x)\cos(x)$.

b) (3 points) $f(x) = \frac{1}{x^2+1} + \frac{1}{x^2-1}$.

c) (2 points) $f(x) = \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}$.

d) (3 points) $f(x) = \log(x) + \frac{1}{\log(x)}$.

Solution:
a) $\sin^6(x)/6 + c$
b) $\arctan(x) + \log(x-1)/2 + \log(x+1)/2 + c$
c) $\arcsin(x) + \arcsin(x)/2 + \sin(2\arcsin(x)))/4$
d) $x\log(x) - x + li(x)$ the second integral is a non-elementary integral. Was a freebe. You got 3 points even without solving that...

Problem 13) Applications (10 points)

a) (5 points) We know the total cost $F(x) = -x^3 + 2x^2 + 4x + 1$ for the quantity x. In order to find the positive **break-even point** x satisfying $f(x) = g(x)$, where $g(x) = F(x)/x$ is the total cost and $f(x) = F'(x)$ is the marginal cost, we do - how sweet it is - find the maximum of the average cost $g(x) = F(x)/x$. Find the maximum!

b) (5 points) We know the **"velocity"**, **"acceleration"** and **"jerk"** as the first second and third derivative of position. The fourth, fifth and sixth derivatives of position as a function of time are called **"snap"**, **"crackle"** and **"pop"** according to characters used in a cereal add. Assume we know the snap $x''''(t) = t$. Find $x(t)$ satisfying $x(0) = x'(0) = x''(0) = 0, x'''(0) = 0$.

Solution:
a) We have to solve the equation $g(x) = 0$ by the strawberry theorem. Giving the equation $-x^2 + 2x + 4 + 1/x = 0$ was enough. The solution needs to be evaluated numerically, for example with Newton.
b) Integrate 4 times to get $x(t) = t^5/120$. All constants are zero.