Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ··· but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th. Do not communicate math during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T (highlighted) F

$\sin(3\pi/2) = -1$

Solution:
Yes,

2) T (highlighted) F

The cotangent function is monotonically decreasing on the open interval $(\pi/4, \pi/2)$.

Solution:
Indeed, its derivative is $1/\cos^2(x)$.

3) T F (highlighted)

The arccot function is monotonically increasing from 1 to 2.

Solution:
It is decreasing.

4) T F (highlighted)

If F is a CDF, then $F(0) = 0$

Solution:
It depends. F(0)=0 only holds if the density $f(x)$ is zero for negative x.

5) T (highlighted) F

$\frac{d}{dx} \log(e^x) = 1$, where as always log is the natural log.

Solution:
First simplify $\log(e^x) = x$. One can also use the chain rule to see it.

6) T (highlighted) F

The limit of $\sqrt{|x|}/\sin(\sqrt{|x|})$ for $x \to 0$ exists and is equal to 1.

Solution:
It is equal to 1 because of the fundamental theorem of trig. Maybe just put $u = \sqrt{|x|}$.

7) | T | F | — T highlighted

If we apply the l'Hospital rule for the limit $\lim_{h \to 0}[f(x+h) - f(x)]/h$ we get $f'(x)$

Solution:
Yes, this is a limit 0/0 and differentiating the top gives $f'(x)$ while differenting the bottom is 1.

8) | T | F | — F highlighted

If $f'(1) = 0$ and $f''(0) = 1$ then f has a local minimum at $x = 0$.

Solution:
We need the first derivative to be zero at 0 not at 1. Yes, this was a bit of a silly catch. It had actually been a typo when writing the problem.

9) | T | F | — T highlighted

The improper integral $\int_{-1}^{1} 1/\sqrt{1-x^2}\, dx$ is finite.

Solution:
Yes, we know the answer is π as we know from the arc-sin distribution.

10) | T | F | — T highlighted

The function $f(x) = 1 + \sin(x^2) - x^3$ has a root in the interval $(-100, 100)$.

Solution:
Use the intermediate value theorem. The function satisfies $f(-100) \geq 100^3 - 2 > 0$ and $f(100) \leq -100^3 + 2 < 0$.

11) | T | F | — F highlighted

If a function f has a local minimum and a local maximum, then it must have a second minimum.

Solution:
A counter example is $f(x) = x^3 - x$. It only has one minimum and one maximum.

12) T **F** To any continuous function f, there exists a unique $F(x)$ such that $F'(x) = f(x)$.

Solution:
F is only determined up to a constant. So, it is not unique.

13) **T** F If you listen to the sound $\log(1+x)\sin(10000x)$, then it gets louder and louder as time goes on.

Solution:
The amplitude grows like $\log(1+x)$. This is the hull function.

14) T **F** The function $f(x) = e^{-x^2}$ has a local minimum at $x = 0$

Solution:
It is a local maximum, not minimum.

15) T **F** The function $f(x) = (x^{25} - 1)/(x^5 - 1)$ has the limit 20 for $x \to 1$.

Solution:
Use Hospital's rule, or heal the function. The limit is $5 = 25/5$, not $20 = 25 - 5$.

16) **T** F If the average cost $F(x)/x$ of an entity is extremal at $x = 2$, then we have a break-even point $f(2) = g(2)$.

Solution:
This is the strawberry theorem.

17) T **F** If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of f.

Solution:
Only if the mean is zero

18) T **F** The Midi function $f(s)$ gives the midi number $f(s)$ as a function of the frequency s.

Solution:
Wrong way. The variable s is the midi number and $f(s)$ is the frequency.

19) **T** F A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

Solution:
True, we have seen that.

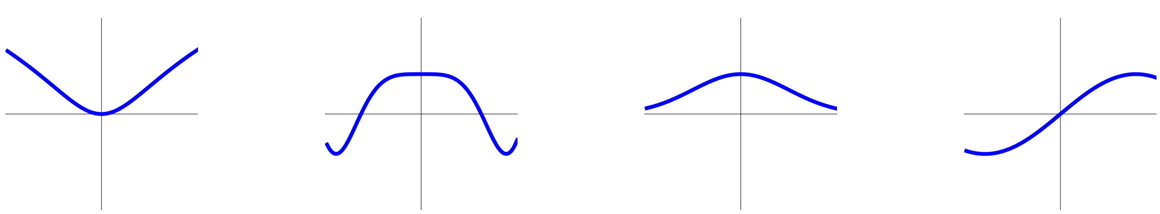
20) **T** F $\sin(\arcsin(1)) = 1$.

Solution:
Yes, the arcsine is the inverse function of sine.

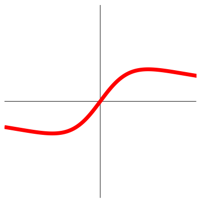
Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

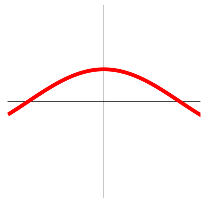
Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)$			
$\cos(x^2)$			
$\log(1+x^2)$			
$\exp(-x^2)$			



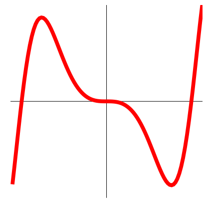
1) 2) 3) 4)



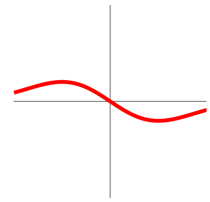
A)



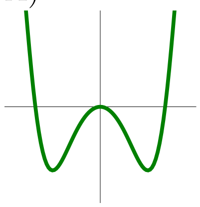
B)



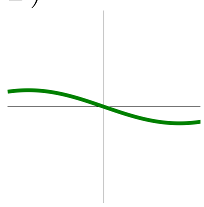
C)



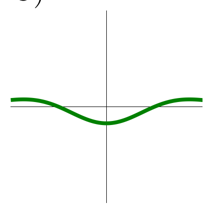
D)



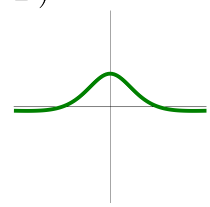
a)



b)



c)



d)

(5 points) Which of the following limits exist in the limit $x \to 1$? If the limit exists, enter the result

Function	Enter the limit if it exists	Check if it does not exist
$\frac{(1-x^9)}{(1-x^7)}$		
$\frac{x}{\log\|x\|}$		
$\frac{\tan(1-x)}{(1-x)}$		
$\log\|x\|/x$		
$\log(x)/\log(2x)$		
$\frac{x^2-1}{\sin(x^2-1)}$		

Solution:
a) 4,B,b
2,C,a
1,A,d
3,D,c
b) 9/7, Limit does not exist, 1,0,0,1

Problem 3) Short answer problem (10 points). No justifications are needed.

a) (3 points) Find the relation between the following functions:

function f	function g	$f = g'$	$g = f'$	none
$\log(x)$	$1/x$			
$1/x$	$-1/x^2$			
$\tan(x)$	$1/(1+x^2)$			
$\cot(x)$	$-1/\sin^2(x)$			
$\arctan(x)$	$1/\cos^2(x)$			
$\text{arccot}(x)$	$-1/(1+x^2)$			

b) (3 points) We integrate $\int_0^1 f(x)\, dx$ numerically. During the numerical integration method lecture we have pointed out that some integration methods give always exact answers for quadratic functions f. For which numerical integration methods is this the case?

Integration method	The method gives the exact value for quadratic f
Archimedes (equal spacing)	
General Riemann sum	
Trapezoid method	
Simpson Method	
Simpson 3/8 Method	

c) (2 points) Formulate the "Strawberry theorem" in economics.

d) (2 points) Which mathematical theorem is involved for the "wobbly table theorem"?

Solution:
a)

function f	function g	$f = g'$	$g = f'$	none
$\log(x)$	$1/x$		x	
$1/x$	$-1/x^2$		x	
$\tan(x)$	$1/(1+x^2)$			x
$\cot(x)$	$-1/\sin^2(x)$		x	
$\arctan(x)$	$1/\cos^2(x)$			x
$\text{arccot}(x)$	$-1/(1+x^2)$		x	

b) Both Simpson's method only.
c) $F' = f, g = F/x$, then $f = g$ is equivalent to $g' = 0$.
d) Intermediate value theorem.

Problem 4) Area computation (10 points)

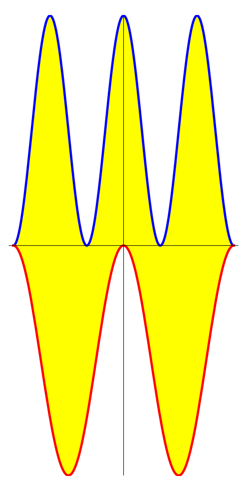
Find the area enclosed by the graphs of the functions

$$f(x) = \cos(2\pi x) - 1 \ .$$

and

$$g(x) = 1 + \cos(3\pi x)$$

on an interval on which f has two minima and g has three maxima. The situation is displayed in the picture.



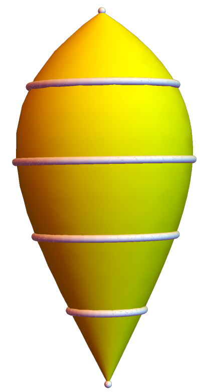
Solution:
The area is $\int_{-1}^{1} (1 + \cos(\pi 3x)) - (\cos(\pi 2x) - 1) dx = 4$.

Problem 5) Volume computation (10 points)

As kids we used to play with a **wooden top**, which is brought in motion by pulling at a rope wound around the solid. We want to the volume of such a top which at height z is a disk of radius

$$r(z) = z(1 - z^3)$$

and where the z values go from 0 to 1.



Solution:
The integral is $\pi \int_0^1 z^2(1+z^3)^2 \, dz = \pi/9$.

Problem 6) Improper integrals (10 points)

Which of the following improper integral are convergent? In each case either state that it is not convergent or compute the limit explicitly.

a) (2 points) $\int_1^\infty \sin(x) \, dx$

b) (2 points) $\int_1^\infty \frac{1}{x^3} \, dx$

c) (2 points) $\int_1^\infty \frac{1}{x^{1/3}} \, dx$

d) (2 points) $\int_0^1 \frac{1}{x^3} \, dx$

e) (2 points) $\int_0^1 \frac{1}{x^{1/3}} \, dx$

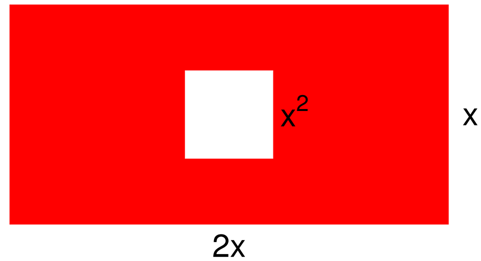
Solution:
a) Does not converge, b) 1/2, c) Does not converge, d) does not converge, e) 3/2

Problem 7) Extrema (10 points)

We want to find the maximal area of a rectangle of length $2x$ and height x in which a square hole of length x^2 has been taken out. The area function is

$$f(x) = 2x^2 - x^4 \ .$$

Use the second derivative test to locate the maximum.



Solution:
Take the derivative $f'(x) = 4x - 4x^3$ and set it to zero. The roots are $0, 1, -1$. We have $f''(1) = -8, f''(-1) = -8, f''(0) = 4$ so that $1, -1$ are maxima and 0 is a minimum. As we deal with a geometric situation $x = 1$ is the solution.

Problem 8) Integration by parts (10 points)

a) (5 points) Compute

$$\int x^5 e^x \ dx \ .$$

b) (5 points) Evaluate the following integral. As always, $\log(x)$ is the natural log.

$$\int \log(x) x \ dx \ .$$

Solution:
a) Use the Tic-Tac-Toe method.

x^5	$\exp(x)$	
$5x^4$	$\exp(x)$	$\oplus$
$20x^3$	$\exp(x)$	$\ominus$
$60x^2$	$\exp(x)$	$\oplus$
$120x$	$\exp(x)$	$\ominus$
120	$\exp(x)$	$\oplus$
0	$\exp(x)$	$\ominus$

The answer is $e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$.
b) Use integration by parts, and differentiate $\log(x)$ and get $x^2 \log(x)/2 - x^2/4 + C$.

Problem 9) Substitution (10 points)

a) (5 points)

$$\int \frac{\sin(\log(x))}{x} \, dx \; .$$

b) (5 points)

$$\int (1-x^2)^{-5/2} dx \; .$$

Solution:
a) With $u = \log(x), du = dx/x$ we get $-\cos(\log(x)) + c$.
b) Use $x = \sin(u), dx = \cos(u)du$ to get $\int \frac{1}{\cos^4(u)} du$. This can be integrated either using the magic box or then by writing $\sec^4(u) = \sec^2(u)\sec^2(u)$ and writing $\sec^2(u) = 1/\cos^2(u) = 1 + \tan^2(u)$ and so get $\frac{1}{\cos^4(u)} = \frac{1}{\cos^2(u)} + \frac{\tan^2(y)}{\cos^2(u)}$ which can be integrated to $\tan(u) + \tan^3(u)/3 + c$. Now back substitute. This turned out to be the trickiest problem in the entire exam.

Problem 10) Advanced integration (10 points)

a) (5 points) Integrate

$$\int \frac{1}{(x+1)(x+3)(x-2)(x-1)} \, dx \; .$$

b) (5 points) Use the magic **trig substitution box** to find the anti-derivative:

$$\int \frac{4dx}{\sin^3(x)} \; .$$

Here is the magic box:

$u = \tan(x/2)$
$dx = \frac{2du}{(1+u^2)}$
$\sin(x) = \frac{2u}{1+u^2}$
$\cos(x) = \frac{1-u^2}{1+u^2}$

Solution:
a) This is a partial fraction method

$$\frac{1}{(x+1)(x+3)(x-2)(x-1)} = \frac{A}{x+1} + \frac{B}{x+3} + \frac{C}{x-2} + \frac{D}{x-1} .$$

We get the constant A for example by multiplying the entire equation gy $x+1$, divide out the $x+1$ factors, then put $x=-1$. This gives $A = 1/12$. Similiarly, the other constants are obtained $B = -1/40, C = 1/15, D = -1/8$. Then, we integrate and get

$$\frac{1}{12}\log(x+1) - \frac{1}{40}\log(x+3) + \frac{1}{15}\log(x-2) - \frac{1}{8}\log(x-1) + C .$$

b) After plugging in the given substitutions, we have

$$4\int 2(1+u^2)^2/8u^3 \, du = \int (1+2u^2+u^4)/u^3 \, du = \int u^{-3}+2/u+u \, du = -1/u^2+2\log(u)+u^2/2+C$$

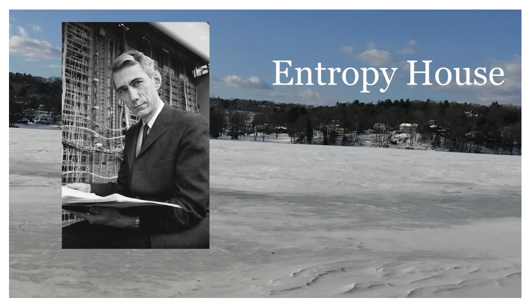
Now back substitute $u = \tan(x/2)$.

Problem 11) PDF's and CDF's. (10 points)

If $f(x)$ is a PDF, then

$$S = -\int_{\infty}^{\infty} f(x)\log(f(x)) \, dx$$

is called the **entropy** of f.
What is the entropy of the exponential distribution, given by the function which is 0 for negative x and e^{-x} for $x \geq 0$?



The Entropy house in Winchester, MA on Mystic Lake, where Claude Shannon, the father of information theory lived. Photo: Oliver Knill, 2018.

Solution:

$$S = -\int_0^\infty f(x)\log(f(x))\,dx = -\int_0^\infty e^{-x}(-x)\,dx$$

This is $\int_0^\infty e^{-x}x\,dx$ which is solved by integration by parts by differentiating $x = u$ and integrating $e^{-x}dx = dv$. The result of this improper integral is 1. We have seen this integral already as the expectation of the exponential distribution.
By the way: This is an important computation. It turns out that from all nice probability distributions on $[0,\infty)$ with mean 1 and variance 1, the exponential distribution is the one which maximizes entropy. On the $(-\infty,\infty)$, among all distributions with mean 0 and variance 1 the normal distribution $f(x) = e^{-x^2/2}/\sqrt{2\pi}$ maximizes entropy.

Problem 12) Which integration method?(10 points)

a) (5 points) Find the anti-derivative of

$$\int e^{e^x} e^x\,dx\ .$$

b) (5 points) And what is the anti-derivative of

$$\int (\log(x))^2 x\,dx\ .$$

Solution:
a) Make a substitution $u = e^x, du = e^x dx$ to get $\int e^u\,du = e^u + c = e^{e^x} + c$.
b) Use Integration by parts $u = \log(x), dv = xdx$ so that we have $\log(x)^2x^2/2 - \int \log(x)x/;dx$. Now use integration by parts again for computing $\int \log(x)xdx$ which is $x^2\log(x)/2 - x^2/4$. The final answer is $x^2\log(x)^2/2 - x^2\log(x)/2 + x^2/4$.

Problem 13) Applications (10 points)

a) (2 points) The CDF of the PDF. What is the CDF of the function that is given as $f(x) = 1/x^2$ if $x \geq 1$ and $f(x) = 0$ else? [Give the function for $x \geq 1$]

b) (2 points) If x is the Midi number, then $f(x) = 440e^{(x-69)/12}$ is called the [give the expression, one word.]

c) (2 points) If $f(x)$ is a PDF, then $M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$ is called a [give the expresion, one word.]

d) (2 points) For the family of functions $f_c(x) = c\cos(x)$, there is a catastrophe at $c =$ [give a number, one number.]

e) (2 points) If $(3, 5)$ and $(2, 7)$ are two data points, the line $y = mx$ which is the best fit minimizes the function [Give a function $f(m) = ...$]

Solution:
a) $F(x) = \int_1^x 1/x^2 = 1 - 1/x$.
b) The frequency.
c) The n'th moment.
d) $c = 0$.
e) $f(m) = (3m - 5)^2 + (2m - 7)^2$.