Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

5/7/2021: Final Exam

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 9 AM on May 8th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F — If f is a PDF, then $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ is called the variance of the distribution.

Solution:
It is $\int_{-\infty}^{\infty} (x-m)^2 f(x)\, dx$ where m is the mean.

2) T F — The function $\log(e^x)$ is defined and continuous for all real numbers x.

Solution:
It simplifies to $\log(e^x) = x$.

3) T F — There exists $p > 0$ such that $\int_0^{\infty} x^p\, dx$ exists and is a finite value.

Solution:
Compare the full problem here. It is the limit $\int_a^b x^p\, dx = b^{p+1}/(p+1) - a^{p+1}/(p+1)$ with $a \to 0$ and $b \to \infty$.

4) T F — $(e^x)^y = (e^y)^x$ for all real numbers x, y.

Solution:
Yes, both are e^{xy}.

5) T F — $e^{(x^y)} = e^{(y^x)}$ for all real numbers x, y.

Solution:
x^y is not the same than y^x.

6) T F — The tangent function is continuous everywhere.

Solution:
It is not continuous at $x = \pi/2$ for example.

7) T F

There was a time in you life when your age $t \in \mathbb{R}$ in units of years was exactly your height $h \in \mathbb{R}$ in units in meters.

Solution:
This follows from the intermediate value theorem. At $t = 0$ you have positive length and zero age and now you have an age which is in meters larger than you are.

8) T F

The limit $\lim_{x \to 0} x \log(|x|)$ is 0.

Solution:
We did that several times. It follows from l'Hopital by writing first $\log |x|/(1/x)$.

9) T F

The Monte-Carlo method is a numerical integration method.

Solution:
Yes, a cool one

10) T F

One of the double angle formulas is $\sin(2x) = \frac{(1-\cos^2(x)}{2}$.

Solution:
Wrong way.

11) T F

The function $f(x) = x(1 - x^5) + \sin(\pi x)$ has a critical point in $(0, 1)$.

Solution:
The derivative is positive at $x = 0$ and negative at $x = 1$ so that there is a point where.

12) T **F** The function $f(x) = \sin(x)/x$ is called the entropy function.

Solution:
It is the sinc function.

13) **T** F If $Df(x) = f(x+1) - f(x)$ and $Sf(x) = f(0) + f(1) + \cdots + f(x-1)$, then $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Solution:
Yes

14) T **F** The function $1/\log(2 - |x|)$ is defined and continuous for all real numbers x.

Solution:
It is not defined at $x = 2$ and $x = -2$.

15) T **F** The Newton iteration method allows to find the roots for any continuous function.

Solution:
The function has to be differentiable in order to apply Newton.

16) **T** F The logarithm function $\log(x)$ is monotonically increasing for all $x > 0$.

Solution:
Yes, the derivative is $1/x$ which is positive for positive x.

17) **T** F Applying the Newton step with the function $f(x) = x^2 - 2$ at the point $x = 1$ gives $T(x) = 1 + 1/2 = 3/2$.

Solution:
yes

18) **T** F The family of functions $f_c(x) = c(x-1)^2 + 3$ experiences a catastrophe at $c = 0$.

Solution:
It is indeed at c=0

19) **T** F If $F(x) = x^2 + 3x$ then $f(x) = 2x + 3$ is called marginal cost.

Solution:
Yes.

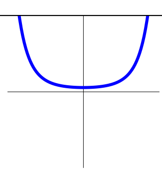
20) **T** F The gradus suavitatis of the fraction 3/8 is equal to 6.

Solution:
Yes, make the factorization $24 = 2*2*2*3$ and apply the formula $1+1+1+1+2$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the name of the functions with their graphs (1-4), with their derivatives (A-D) (middle row) and with the second derivatives (a-d) (last row). If you write on your own paper, please copy the table first.

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$1/(1+x^2)$			
$\sin(x^3)$			
$x^3 - x$			
$\frac{e^{1+x^2}}{20}$			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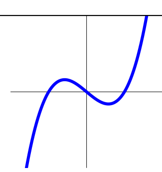
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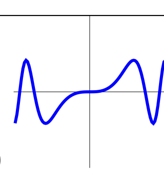
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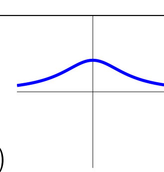
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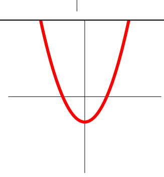
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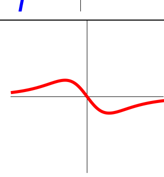
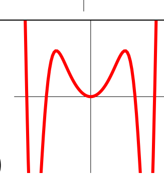

B)

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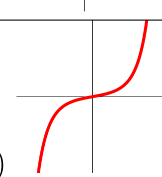
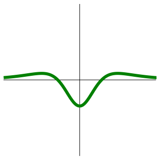
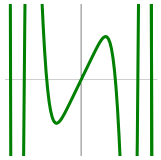
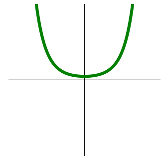
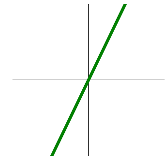

D)

a)  b)  c)  d) 

(5 points) In which cases can we take the limit $x \to 0$? If there is a limit, enter it in the left column, otherwise cross check the right column. If you write on your own paper, please copy the table first.

Function	The limit is (if it exists)	Cross check if not existing
$\frac{\sin(17x)}{\sin(23x)}$		
$-x \log \lvert 3x \rvert$		
$\frac{\sin(x^2)}{\sin^2(x)}$		
$\log \lvert 5x \rvert / \log \lvert 7x \rvert$		
$\arctan(x)/\tan(x)$		
$\frac{\cos(x)+1}{x^2}$		

Solution:
a)
4 B a
3 C b
2 A d
1 D c
b) 17/23,0,1,1,1,DNE

Problem 3) Short answer problem (10 points). No justifications are needed.

a) (3 points) Complete the following table of probability distributions and cumulative distribution functions.

PDF	PDF supported on	CDF on that interval
e^{-x}	$[0,\infty)$	
	$(-\pi/2,\pi/2)$	$\frac{\arcsin(x)}{\pi}+\frac{1}{2}$
	$(-\infty,\infty)$	$\frac{\arctan(x)}{\pi}+\frac{1}{2}$

b) (3 points) We integrate $\int_0^1 x\,dx$ numerically, using only $n=1$ interval and compare three different integration methods. What values do we get in each case?

Integration method	Which value does the method give?
Trapezoid method	
Simpson Method	
Simpson 3/8 Method	

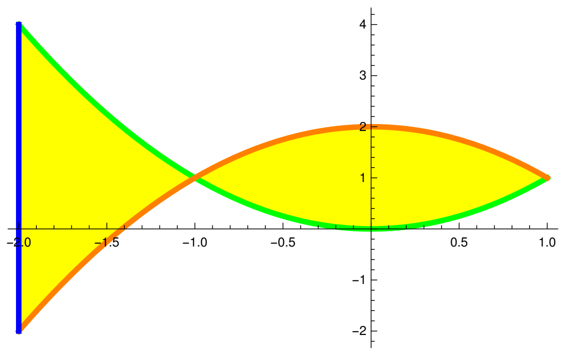
c) (2 points) If f is the marginal cost and F the total cost and g the average cost. What is the definition of the **break even point** in this context?

d) (2 points) What theorem is responsible for the fact that there is a point on earth such that the temperature on P and its anti-pod point Q are exactly the same?

Solution:
a) $1-e^{-x}$, $(1-x^2)^{-1/2}/\pi$, $(1+x^2)^{-1}/\pi$ b) 1/2,1/2,1/2
c) f=g
d) Intermediate value theorem

Problem 4) Area computation (10 points)

What is the area of the **fish shape** region bordered by the curves $y = x^2, y = 2-x^2, x = -2$? We stress that we want the **area** of the region and **not** the signed area of the region.



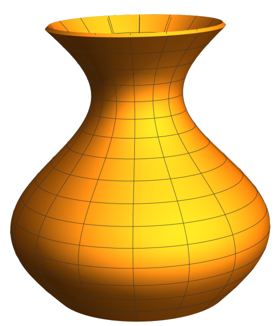
Solution:
Split up the integral $\int_{-2}^{-1} (2x^2 - 2)\ dx + \int_{-1}^{1} (2 - 2x^2)\ dx = 16/3$.

Problem 5) Volume computation (10 points)

Compute the volume of the **vase** which has at height z a **circular cross sections of radius**

$$r(z) = 2 + \sin(z)$$

and where the height z is in the interval $0 \leq z \leq 2\pi$.



Solution:
Use the double angle formula. $\pi \int_0^{2\pi} (2 + \sin(z))^2\ dz = 9\pi^2$.

Problem 6) Improper integrals (10 points)

You see here five improper integrals. Pick the ones which converge and compute their value. For the others, state that the integral does not exist and give a short explanation.

a) (2 points) $\int_0^1 x^{-6}\, dx$

b) (2 points) $\int_0^1 x^{-1/6}\, dx$

c) (2 points) $\int_1^\infty x^{-6}\, dx$

d) (2 points) $\int_1^\infty x^{-1/6}\, dx$

e) (2 points) $\int_0^\infty x^{-6}\, dx$.

Solution:
In the first two cases write it as a limit $\lim_{a\to 0}^1 f(x)dx$, in the next two cases, write it as a limit $\lim_1^{b\to\infty} f(x)dx$ In the last case $\lim_{a\to 0}\lim_{b\to\infty}\int_a^b f(x)dx$. DNE, 6/5,1/5, DNE, DNE.

Problem 7) Extrema (10 points)

A **trapezoid** of height $2x$, bottom length x and top length $(4-2x^2)$ has area

$$f(x) = x(x + (4-2x^2)) = 4x + x^2 - 2x^3 \ .$$

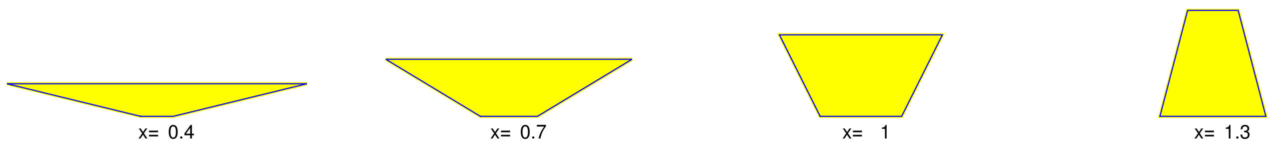
The figure below illustrates the situation for some x values.

a) (3 points) For which x does f have a local maximum? Use the second derivative test to check.

b) (3 points) On which interval $[a, b]$ does the problem make sense (height, lengths and area can not be negative)? What are the function values on the boundary?

c) (2 points) There a theorem which assures that $f(x)$ has a global maximum on that interval $[a, b]$? What is the name of the theorem?

d) (2 points) What is the global minimum of $f(x)$ on the interval of consideration?



Solution:
$f'(x) = 4 + 2x - 6x^2$ for $x = 1$ and $x = -2/3$. Only $x = 1$ makes sense in this set-up. The function is 0 at $x = 0$ and $x = (1 + \sqrt{33})/4$ but the function is only defined on $[0, \sqrt{2}]$, because $4 - 2x^2$ has to be positive. The minimum is at $x = 0$. The point $x = 1$ is a maximum because $f''(x) = 2 - 12x$ is -10 at $x = 1$. The extremal value theorem assures that we have a maximum.

Problem 8) Integration by parts (10 points)

a) (5 points) Compute the following anti-derivative:

$$\int (x - 2)^4 e^{x/2} \, dx \, .$$

b) (5 points) And now have some fun riding the "merry go round" for the following integral

$$\int e^x \sin(x)\ dx\ .$$

Solution:
a) Use the Tic-Tac-Toe method.

$(x-2)^4$	$\exp(x/2)$	
$4(x-2)^3$	$2\exp(x/2)$	$\oplus$
$12(x-2)^2$	$4\exp(x/2)$	$\ominus$
$24(x-2)$	$8\exp(x/2)$	$\oplus$
24	$16\exp(x/2)$	$\ominus$
0	$32\exp(x/2)$	$\ominus$

and add up.
b) $(e^x \sin(x) - e^x \cos(x))/2 + C$.

Problem 9) Substitution/chain rule (10 points)

a) (5 points)

$$\int \cos(\cos(\cos(x)))\sin(\cos(x))\sin(x)\ dx\ .$$

b) (5 points)

$$\int \frac{2\log(\log(\log(x)))}{\log(\log(x))\log(x)x} dx\ .$$

Solution:
a) $\sin\cos(\cos(x))) + C$
b) $\log^2(\log(\log(x))) + C$.

Problem 10) Advanced integration (10 points)

a) (5 points) Integrate

$$\int \frac{1}{(x+8)(x+4)(x+2)}\, dx \; .$$

b) (5 points) Use the magic **trig substitution box** to find the anti-derivative:

$$\int \frac{1}{\cos(x)} + \frac{1}{\sin(x)} dx \; .$$

Here is the magic box:

$u = \tan(x/2)$
$dx = \frac{2du}{(1+u^2)}$
$\sin(x) = \frac{2u}{1+u^2}$
$\cos(x) = \frac{1-u^2}{1+u^2}$

Solution:
a) Use the residue method to rewrite the integrand as $\log(x+8)/24 - \log(x+4)/8 + \log(x+2)/12$ Then integrate.
b) $\int 2/(1-u^2) du + 1/u \; du$. Now back substitute.

Problem 11) PDF's and CDF's. (10 points)

We look at two functions which are defined to be zero outside the interval $[0, \pi]$ and which are given on $0 \leq x \leq \pi$ as

$$f(x) = \frac{\sin(x)}{2}$$

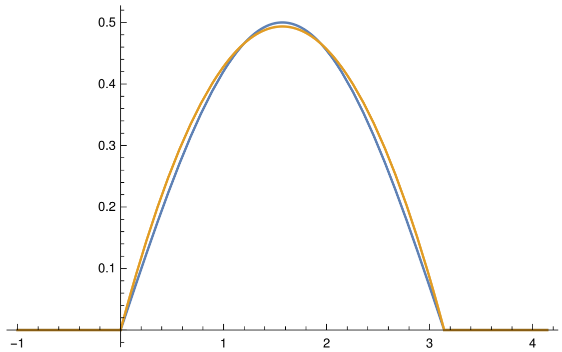
and

$$g(x) = \frac{x(\pi - x)}{5} \; .$$

Their graphs are very close to each other, as you can see in the picture.

a) (5 points) One of these functions produces a probability distribution. Which one? Justify your answer.

b) (5 points) The other function needs to be multiplied with a constant c to become a probability distribution. What is this constant?



Solution:
f is a probability distribution. The second total integral is $\pi^3/30$. The constant is $30/\pi^3$.

Problem 12) Which integration method? (10 points)

a) (5 points) Determine the value of the definite integral

$$\int_0^{\pi/2} e^{\sin(x)} \cos(x) \, dx \; .$$

b) (5 points) Find the anti-derivative

$$\int 3\sqrt{1-x^2} + \frac{4}{\sqrt{1-x^2}} \, dx \; .$$

Solution:
a) $e - 1$.
b) The second integral is $4 \arcsin(x)$. The first is done with a trig substitution $x = \sin(u)$, to get $3 \int \cos^2(u)\, du$ which can be solved by an integration formula.

Problem 13) Applications (10 points) Only answers are needed

a) (2 points) What is the expectation $\int x f(x)\, dx$ of the probability distribution function $f(x) = e^{-|x|}/2$ which is a function defined for all real x. [We only need a number.]

b) (2 points) What is the inverse function of the (fictional) **Stockhausen Midi function** $f(x) = 440 \cdot 5^{(x-25)/5}$? This function gives the midi number x as a function of the frequency f.

c) (2 points) Which digit among $0, 1, \dots, 8, 9$ is the **least frequent** in **Benford's distribution law**? [You only need to provide the digit]

d) (2 points) The family of functions $f_c(x) = x^4 - cx^2 + x^2$ experiences a **catastrophe**. What is the parameter c for which this happens? [You only need to find that parameter.]

e) (2 points) If $(-2, -7)$ and $(2, 7)$ are two data points. What is the best linear fit $y = ax + b$ which passes through the two points?

Solution:
a) 0, $x = 25 + 5 \log_5(f/440)$, 9, $c = 1$, and $y = (7/2)x$.