Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2020

4/8/2020: First hourly Practice E

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. Hand-write your paper. It can be handwritten also electronically using an ipad but it needs to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ 20: False. Also, please sign your solutions.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper.
- The exam will be released at 9 AM on April 8. Try to do it during regular class time. You have to submit the paper within 24 hours by emailing it to knill@math.harvard.edu as an attachment. Submit it soon so that technical problems can be ironed out. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F

The formula $\int_0^x f''(x)\ dx = f'(x) - f'(0)$ holds.

Solution:
Apply the fundamental theorem to the derivative.

2) T F

The area of the lower half disc is the integral $\int_{-1}^{1} -\sqrt{1-x^2}\ dx$

Solution:
The area is positive. The integral given is negative.

3) T F

If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ we obtain a solid with volume $\int_0^1 \pi x^4\ dx$.

Solution:
Indeed the area is $A(x) = \pi x^4$.

4) T F

The identity $d/dx \int_0^x f''(t)\ dt = f'(x)$ holds.

Solution:
The result is $f''(x)$.

5) T F

There is a point in $[0,1]$, where $f'(x) = 0$ if $f(x) = x^3 - x^2 + 1$.

Solution:
Since $f(0) = f(1) = 1$, Rolle's theorem assures this.

6) T F

The fundamental theorem of calculus assures that $\int_a^b f'(x)\ dx = f(b) - f(a)$.

Solution:
Yes, this is one of the important reformulations.

7) T F If f is differentiable on $[a, b]$, then $\int_a^b f(x)\, dx$ exists.

Solution:
Yes, we have seen the proof in class.

8) T F The integral $\int_0^{\pi/2} \sin(\sin(x))\, dx$ is positive.

Solution:
$\sin(\sin(x)) > 0$ there so that the integral is positive

9) T F The anti-derivative of an anti-derivative of f is equal to the derivative of f.

Solution:
This is just total nonsense. We would have to differentiate three times the anti derivative of the anti derivative to get to the derivative of f.

10) T F If a function is positive everywhere, then $\int_a^b f(x)\, dx$ is positive too.

Solution:
Yes, the integral has the meaning of an area under the curve.

11) T F If a differentiable function is odd, then $\int_{-1}^1 f(x)\, dx = 0$.

Solution:
We have a cancellation to the left and to the right.

12) T F If $f_c(x)$ is a function with a local minimum at 0 for all $c < 0$ and no local minimum in $[-1, 1]$ for $c > 0$, then $c = 0$ is called a catastrophe.

Solution:
Yes this is a pretty precise definition of a catastrophe.

13) T **F** The term ”improper integral” is a synonym for ”indefinite integral”.

Solution:
Improper means that we either integrate a function which has a discontinuity or that we integrate over an infinite interval.

14) T **F** The function $F(x) = x\sin(x)$ is an antiderivative of $\sin(x)$.

Solution:
Too good to be true. Just differentiate and you see that the derivative of F is not f.

15) T **F** The mean value theorem holds for every continuous function.

Solution:
No, only for differentiable functions.

16) T **F** Newton and Leibniz were best buddies all their life. Leibniz even gave once the following famous speech: ”You guys might not know this, but I consider myself a bit of a loner. I tend to think of myself as a one-man wolf pack. But when my sister brought Isaac home, I knew he was one of my own. And my wolf pack ... it grew by one.

Solution:
This line is from the movie ”hangover”. No, Newton and Leibniz had a dispute about who discovered calculus.

17) T **F** Any function $f(x)$ satisfying $f(x) > 0$ is a probability density function.

Solution:
What is missing is that the integral over the real line is equal to 1.

18) **T** F The moment of inertia integral I can be used to compute energy with the relation $E = \omega^2 I/2$ where ω is the angular velocity.

Solution:
Yes, this is why the moment of inertia is an important quantity in physics.

19) | T | F | If $0 \leq f(x) \leq g(x)$ then $0 \leq \int_0^1 f(x)\, dx \leq \int_0^1 g(x)\, dx$.

Solution:
Yes, just look at the areas.

20) | T | F | The improper integral $\int_0^\infty 1/(x^4+1)\, dx$ is finite.

Solution:
It works well at 0 because the function does not have a pole there. The integral is smaller than $\int 1/x^4\, dx$ which has the anti-derivative $-1/(3x^3)$. The improper integral exists.

Problem 2) Matching problem (10 points) No justifications are needed.

From the following functions there are two for which no elementary integral is found. Find them. You can find them by spotting the complement set of functions which you can integrate.

Function	Antiderivative is not elementary	Function	Antiderivative is not elementary
e^{-x^2}		$1/\log(x)$	
$\sin(3x)$		$\tan(3x)$	
$1/x$		$\arctan(3x)$	

Solution:
The $1/\log(x)$ and e^{-x^2} have no elementary integral.

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following functions are PDF's, which are CDF's, which are neither?

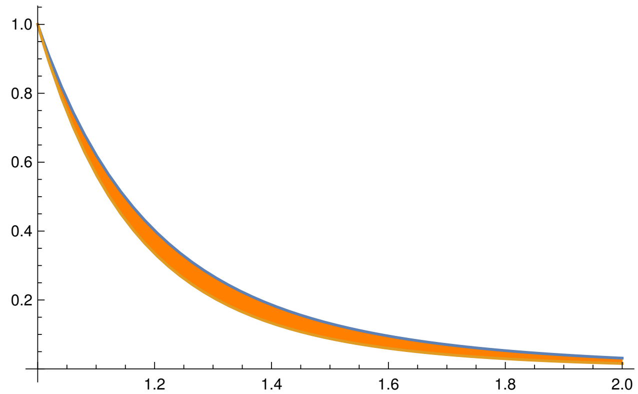
Function	PDF	CDF
$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$		
$\mathrm{Erf}(x)$		
$f(x) = e^{-\lvert x\rvert}/2$		
$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$		
$f(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$		

Solution:
PDF, CDF,PDF,PDF,PDF

Problem 4) Area computation (10 points)

a) (5 points) Find the area of the region enclosed by the curves $3 - x^4$ and $3x^2 - 1$.

b) (5 points) Find the area of the region between $1/x^6$ and $1/x^5$ from $x = 1$ to $x = \infty$.

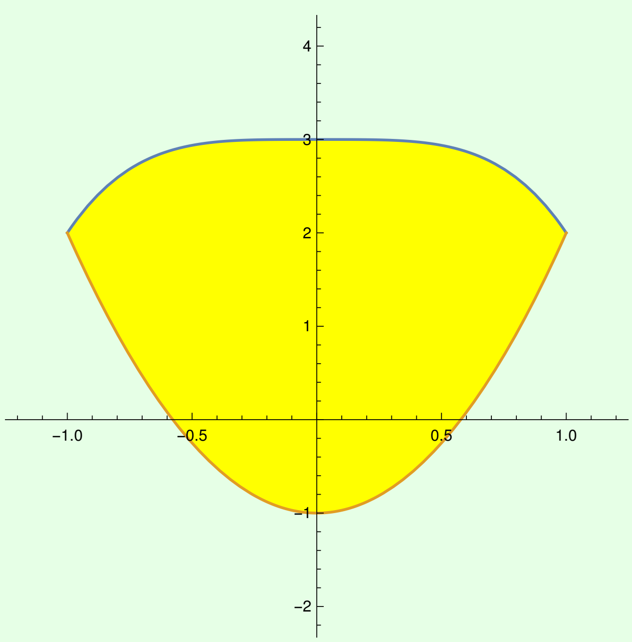


Solution:
a) The two curves intersect at $x = -1$ and $x = 1$ so that the area is

$$\int_{-1}^{1} (3 - x^4) - (3x^2 - 1)\, dx = 3x - x^5/5 - x^3 + x|_{-1}^{1} = 28/5 \; .$$

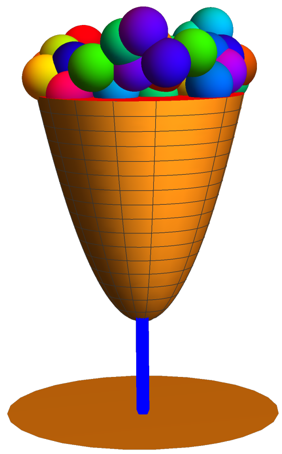
b) The graph of $1/x^5$ is above $1/x^6$. We have

$$\int_{1}^{\infty} x^{-5} - x^{-6}\, dx = -x^{-4}/4 + x^{-5}/5|_{1}^{\infty} = \frac{1}{4} - \frac{1}{5} = \frac{1}{20} \; .$$



Problem 5) Volume computation (10 points)

Cody eats some magic "Bertie Botts Every Flavor Beans" from a cup which is a rotationally symmetric solid, for which the radius at position x is $\sqrt{x}$ and $0 \leq x \leq 4$. Find the volume of Cody's candy cup.



Solution:
The area at height x is $\sqrt{x}^2 \pi = x\pi$ so that the volume is

$$\int_0^4 x\pi \, dx = \pi x^2/2|_0^4 = 8\pi \; .$$

Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x + \tan(x) + \sin(x) + \cos(x) + \log(x) \, dx$.

b) (5 points) $\int_1^3 (x+1)^3 \, dx$

Solution:
a) The anti-derivative is $x^2/2 - \log(\cos(x)) - \cos(x) + \sin(x) + (x\log(x) - x)$. (The $\log(x)$ integral is a bit out of line at this stage, but we have seen it at some point).
b) $(x+1)^4/4|_1^3 = (4^4 - 2^4)/4 = 60$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives

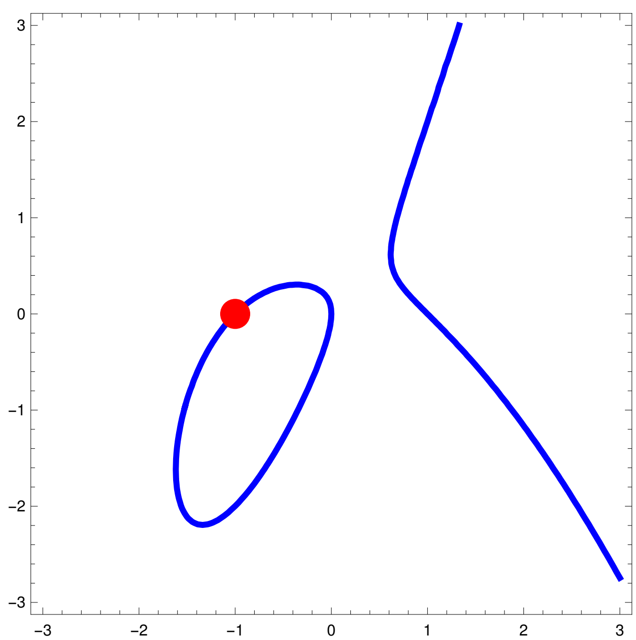
a) (5 points) $\int \sqrt{x^3}\, dx$

b) (5 points) $\int 4/\sqrt{x^5}\, dx$

Solution:
a) The function is $x^{3/2}$. Integration gives $x^{5/2}(2/5) + C$.
b) The function is $4x^{-5/2}$. Integration gives $4x^{-3/2}(-2/3) + C$.

Problem 8) Implicit differentiation (10 points)

The curve $y^2 = x^3 + 2xy - x$ is an example of an **elliptic curve**. Find dy/dx at the point $(-1, 0)$ without solving for y first.



Solution:
Differentiate, $2yy' = 3x^2 + 2y + 2xy' - 1$ and solve for $y' = (3x^2 + 2y - 1)/(2y - 2x)$. At the point $(-1, 0)$ this is 1.

Problem 9) Applications (10 points)

The probability density of the exponential distribution is given by $f(x) = (1/2)e^{-x/2}$. The probability to wait for for time x (hours) to get an idea for a good calculus exam problem is $\int_0^x f(x)\, dx$. What is the probability to get a good idea if we wait for $T = 10$ (hours)?

Solution:
$\int_0^{10} (1/2)e^{-x/2}\, dx = (1 - e^{-5})$ which is almost certain. Indeed, we did not have to wait so long to get this great problem.

Problem 10) Applications (10 points)

What is the **average value** of the function

$$f(x) = 4 + 1/(1 + x^2)$$

on the interval $[-1, 1]$?

Solution:
It is $\int_{-1}^{1} f(x)\, dx/2 = (8 - 2\arctan(1))/2 = 4 + \arctan(1) = 4 + \pi/4$.