

PROBABILITY THEORY

MATH 154

Take home midterm, Due 3/6/26, midnight

Handwritten. No web-search, no external help, no correspondence, no computer algebra system, no AI. Once you started reading this, no sources, and no notes can be used before submission. You can use a personally handwritten page written by yourself before the midterm. No discussion about the exam before the submission date 3/6/26 midnight. Write your name on each page and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."**

Problem A (10 points): In this problem, we assume that $(\Omega, \mathcal{A}, P)$ is an arbitrary probability space.

- a) Assume that A, B are independent elements in $\mathcal{A}$. Prove that $P[A + B]$ is determined from $a = P[A]$ and $b = P[B]$ by giving a formula of $P[A + B]$ that only involves a, b .
- b) Assume that A, B are independent elements in $\mathcal{A}$. Show that $P[A \setminus B]$ is determined from $a = P[A]$ and $b = P[B]$.
- c) Assume A, B are disjoint independent elements in $\mathcal{A}$. Prove that either A or B must be the empty set (up to measure zero).
- d) Verify that if $\{A, B, C\}$ are independent sets, then C is independent of $A + B$. Also show that C is independent of $A \cdot B$.
- e) Assume that A_n is a sequence of events that satisfy $\sum_{n=1}^{\infty} P[A_n] = 2026$. What is $P[\bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n]$?

Problem B (10 points): In this problem a) and b) belong together. Random variables in c), d), e) are each unrelated to any of the others.

- a) Assume the CDF of a random variable X is $F_X(t) = 1 - e^{-5t}$ for $t \geq 0$ and $F_X(t) = 0$ otherwise. What is the PDF $f_X(t)$?
- b) What is the expectation and variance of X with the CDF in a)?
- c) The **characteristic function** of a random variable X is $\cos(t)$, the characteristic function of Y is $\cos(2t)$. Assuming X, Y are independent. What is the characteristic function of $X + Y$?
- d) There is a general formula that determines the CDF $F_X(t)$ of X from $\phi_X(t)$. State it.
- e) What is the 4th moment $\mathbb{E}[Z^4]$ of the random variable Z with characteristic function $\phi_Z(t) = \cos(3t)$?

Problem C (10 points): Let $([0, 1] \times [0, 1], \mathcal{A}, P)$ be the probability space on the unit square Ω . The probability measure satisfies $P[[a, b) \times [c, d)] = (b-a)(d-c)$ on the π -system of all half open squares. Define the events $A = [1/2, 1] \times [0, 1]$, $B = [0, 1] \times [0, 1/3]$ and the random variables $X(x, y) = x^2$, $Y(x, y) = y$.

- Compute $P[B|A]$, $P[A|B]$ and verify the **Bayes law** in this case.
- Compute the **expectation**, and **variance** of both X and Y .
- Find the **covariance** $\text{Cov}[X, Y]$ and **correlation** $\text{Corr}[X, Y]$.
- Explain why A is in the σ -algebra $\mathcal{A}_X$ generated by X and why B is in the σ algebra $\mathcal{A}_Y$ generated by Y .
- Explain why X and Y are independent.

Problem D (10 points): In this problem, a, b, λ are positive constants. If X is a Cauchy distributed random variable with probability density $f_X(x) = \frac{1}{\pi(1+\frac{x^2}{a^2})}$, then a computation shows that the characteristic function is $\phi_X(t) = e^{-a|t|}$. The **standard Cauchy distribution** is $a = 1$.

- Write down the integral which needs to be checked in order to verify that $\phi_X(t)$ is indeed the characteristic function of $f_X(x)$. You do not have to evaluate this integral.
- Assume Y is a second Cauchy distributed random variable with probability density $f_Y(x) = \frac{1}{\pi(1+\frac{x^2}{b^2})}$. Assuming that X, Y are independent, argue carefully why $X + Y$ is a Cauchy distributed random variable.
- Prove that if X is Cauchy distributed, then $Y = \lambda X$ is Cauchy distributed if $\lambda > 0$.
- How do the characteristic functions of X and λX relate?
- Verify that if X, Y are standard Cauchy distributed random variables that are independent, then $(X + Y)/2$ is a standard Cauchy distributed random variable.

Problem E (10 points): We look at some Paradoxa and Theorems:

- What did the **Bertrand paradox** show us? State content and significance.
- State the content of the **Carathéodory theorem**.
- What structure is the smallest π -system that contains a λ -system $\mathcal{A}$.
- Explain the Simpsons paradox in words.
- We verify a more sophisticated **Bayes formula**. Recall that a partition of Ω is a disjoint union of events, whose union is Ω . Prove the following: "Given a finite partition $\{A_1, \dots, A_n\}$ in $\mathcal{A}$ and $B \in \mathcal{A}$ with $P[B] > 0$, then for every $1 \leq k \leq n$, we have

$$P[A_k|B] = \frac{P[B|A_k]P[A_k]}{\sum_{j=1}^n P[B|A_j]P[A_j]}.$$