

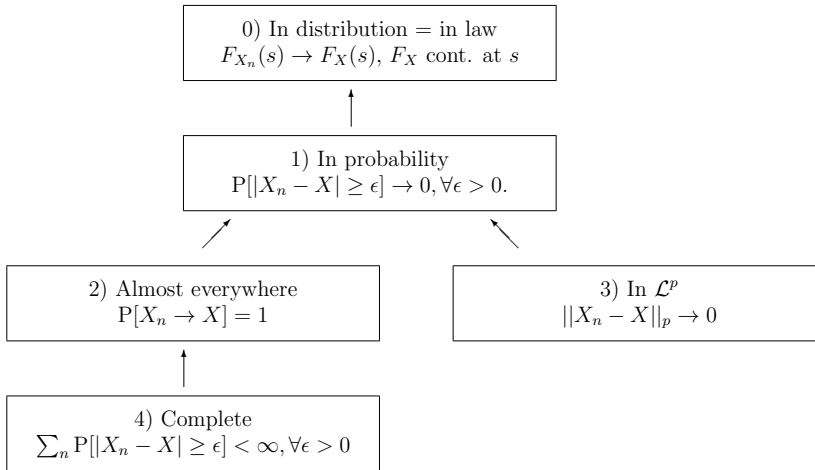
PROBABILITY THEORY

MATH 154

Unit 13: Stochastic convergence

13.1. A sequence of random variables X_n on $(\Omega, \mathcal{A}, P)$ is called a **stochastic process**. We say X_n converges **in probability** to X , if $P[|X_n - X| \geq \epsilon] \rightarrow 0$ for all $\epsilon > 0$. We say X_n converges **almost everywhere** or **almost surely** to X if $P[X_n \rightarrow X] = 1$. We say X_n **converges in $\mathcal{L}^p$** to X , if $\|X_n - X\|_p \rightarrow 0$ for $n \rightarrow \infty$. Finally, X_n converges **completely** if $\sum_n P[|X_n - X| \geq \epsilon] < \infty$ for all $\epsilon > 0$.

13.2. A sequence X_n of random variables X_n on $(\Omega_n, \mathcal{A}_n, P_n)$ converges **in distribution**, if $F_{X_n}(s) \rightarrow F_X(s)$ at all continuity points s of F_X . We say X_n converges **in law** to X , if the laws μ_n of X_n converge weakly to the law μ of X meaning that for every continuous function f on $\mathbb{R}$ of compact support, one has $\int f(x) d\mu_n(x) \rightarrow \int f(x) d\mu(x)$.



Proof. $[2) \Rightarrow 1)$: $\{X_n \rightarrow X\} = \bigcap_k \bigcup_m \bigcap_{n \geq m} \{|X_n - X| \leq 1/k\}$ is equivalent to $1 = P[\bigcup_m \bigcap_{n \geq m} \{|X_n - X| \leq \frac{1}{k}\}] = \lim_{m \rightarrow \infty} P[\bigcap_{n \geq m} \{|X_n - X| \leq \frac{1}{k}\}]$ for all k and so $0 = \lim_{n \rightarrow \infty} P[\bigcup_{n \geq m} \{|X_n - X| \geq \frac{1}{k}\}]$ for all k . Therefore $P[|X_m - X| \geq \epsilon] \leq P[\bigcup_{n \geq m} \{|X_n - X| \geq \epsilon\}] \rightarrow 0$ for all $\epsilon > 0$. $[4) \Rightarrow 2)$: The first Borel-Cantelli lemma implies that for all $\epsilon > 0$ $P[\{|X_n - X| \geq \epsilon, \text{ infinitely often}\}] = 0$. We get so for $\epsilon_k \rightarrow 0$ the relation $P[B_k] = P[\bigcup_n \{|X_n - X| \geq \epsilon_k, \text{ infinitely often}\}] \leq \sum_n P[\{|X_n - X| \geq \epsilon_k, \text{ infinitely often}\}] = 0$ and $\bigcup_k B_k$ has measure zero. Now $P[X_n \rightarrow X] = 1 - P[\bigcup_k B_k] = 1 - 0 = 1$. $[3) \Rightarrow 1)$: Chebychev-Markov implies $P[|X_n - X| \geq \epsilon] \leq \frac{E[|X_n - X|^p]}{\epsilon^p}$. $[1) \Rightarrow 0)$: $P[X_n \leq c] \leq P[X \leq c + \epsilon] + P[|X_n - X| > \epsilon]$. $\square$

Theorem 1. Given $X_n \in \mathcal{L}^\infty$ with $\|X_n\|_\infty \leq K$ for all n , then $X_n \rightarrow X$ in probability if and only if $X_n \rightarrow X$ in $\mathcal{L}^1$.

Proof. (i) For $k \in \mathbb{N}$, $P[|X| > K + \frac{1}{k}] \leq P[|X - X_n| > \frac{1}{k}] \rightarrow 0, n \rightarrow \infty$ so that $P[|X| > K + \frac{1}{k}] = 0$. Therefore $P[|X| > K] = P[\bigcup_k \{|X| > K + \frac{1}{k}\}] = 0$. (ii) Given $\epsilon > 0$. Choose m such that $P[|X_n - X| > \frac{\epsilon}{3}] < \frac{\epsilon}{3K}$ for all $n > m$. Use the notation $E[X; A] = E[X \cdot 1_A]$. By (i) we have $E[|X_n - X|] = E[(|X_n - X|; |X_n - X| > \frac{\epsilon}{3}) + E[(|X_n - X|; |X_n - X| \leq \frac{\epsilon}{3})] \leq 2KP[|X_n - X| > \frac{\epsilon}{3}] + \frac{\epsilon}{3} \leq \epsilon$. $\square$

13.3. A family $\mathcal{C} \subset \mathcal{L}^1$ of random variables is called **uniformly integrable**, if

$$\lim_{R \rightarrow \infty} \sup_{X \in \mathcal{C}} E[X; |X| > R] = 0$$

for all $X \in \mathcal{C}$ still using notation $E[X; A] = E[X 1_A]$.

Theorem 2. Given $X \in \mathcal{L}^1$ and $\epsilon > 0$. There exists $K \geq 0$ with $E[|X|; |X| > K] < \epsilon$.

Proof. Assume we are given $\epsilon > 0$. If $X \in \mathcal{L}^1$, we can find $\delta > 0$ such that if $P[A] < \delta$, then $E[|X|; A] < \epsilon$. Since $KP[|X| > K] \leq E[|X|]$, we can choose K such that $P[|X| > K] < \delta$. Therefore $E[|X|; |X| > K] < \epsilon$. $\square$

13.4. The next proposition gives a necessary and sufficient condition for $\mathcal{L}^1$ convergence.

Theorem 3. Given $X_n \in \mathcal{L}^1$ and $X \in \mathcal{L}^1$. The following is equivalent:

- a) X_n converges in probability to X and $\{X_n\}_{n \in \mathbb{N}}$ is uniformly integrable.
- b) X_n converges in $\mathcal{L}^1$ to X .

Proof. a) $\Rightarrow$ b). For any random variable X and $K \geq 0$ define the bounded variable $X^{(K)} = X \cdot 1_{\{-K \leq X \leq K\}} + K \cdot 1_{\{X > K\}} - K \cdot 1_{\{X < -K\}}$. By the uniform integrability condition and the previous theorem applied to $X^{(K)}$ and X , we can choose K such that for all n , $E[|X_n^{(K)} - X_n|] < \frac{\epsilon}{3}$, $E[|X^{(K)} - X|] < \frac{\epsilon}{3}$. Since $|X_n^{(K)} - X^{(K)}| \leq |X_n - X|$, we have $X_n^{(K)} \rightarrow X^{(K)}$ in probability. By the last theorem we know $X_n^{(K)} \rightarrow X^{(K)}$ in $\mathcal{L}^1$ so that for $n > m$ $E[|X_n^{(K)} - X^{(K)}|] \leq \epsilon/3$. Therefore, for $n > m$ also

$$E[|X_n - X|] \leq E[|X_n - X_n^{(K)}|] + E[|X_n^{(K)} - X^{(K)}|] + E[|X^{(K)} - X|] \leq \epsilon.$$

b) $\Rightarrow$ a). We have seen already that $X_n \rightarrow X$ in probability if $\|X_n - X\|_1 \rightarrow 0$. We have to show that $X_n \rightarrow X$ in $\mathcal{L}^1$ implies that X_n is uniformly integrable.

Given $\epsilon > 0$. There exists m such that $E[|X_n - X|] < \epsilon/2$ for $n > m$. By the absolute continuity property, we can choose $\delta > 0$ such that $P[A] < \delta$ implies

$$E[|X_n|; A] < \epsilon, 1 \leq n \leq m, E[|X|; A] < \epsilon/2.$$

Because X_n is bounded in $\mathcal{L}^1$, we can choose K such that $K^{-1} \sup_n E[|X_n|] < \delta$ which implies $P[|X_n| > K] < \delta$. For $n \geq m$, we have therefore, using the notation $E[X; A] = E[X \cdot 1_A]$

$$E[|X_n|; |X_n| > K] \leq E[|X|; |X_n| > K] + E[|X - X_n|] < \epsilon.$$

$\square$