PROBABILITY THEORY

MATH 154

Homework 10

Random walks and Markov

Problem 10.1: a) Compute the equilibrium distribution for the Markov matrix $A = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$.
b) Compute the equilibrium distribution in the example $\begin{bmatrix} 5/6 & 1/6 & 0 \\ 0 & 2/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \end{bmatrix}$.

Problem 10.2: A **stochastic matrix** or **Markov Matrix** is a matrix, such that every column is a probability vector.
a) A Markov matrix always has an eigenvalue 1. Its eigenvector is the equilibrium distribution. Conclude that if A is a Markov matrix such that A^n is positive for some n, then A has a unique equilibrium distribution.
b) How can you apply the Frobenius theorem to see that for any connected graph, there exists a unique equilibrium measure for the symmetric random walk. (For simplicity, you can assume the graph is also not bipartite as in the bipartite case, A^n is not necessarily positive for some n).

Problem 10.3: Solve the Good Will Hunting problem for the graph given in the picture.
1) the adjacency matrix A.
2) the matrix giving the number of 3 step walks.
3) the generating function for walks from point $i \to j$.
4) the generating function for walks from points $1 \to 3$.

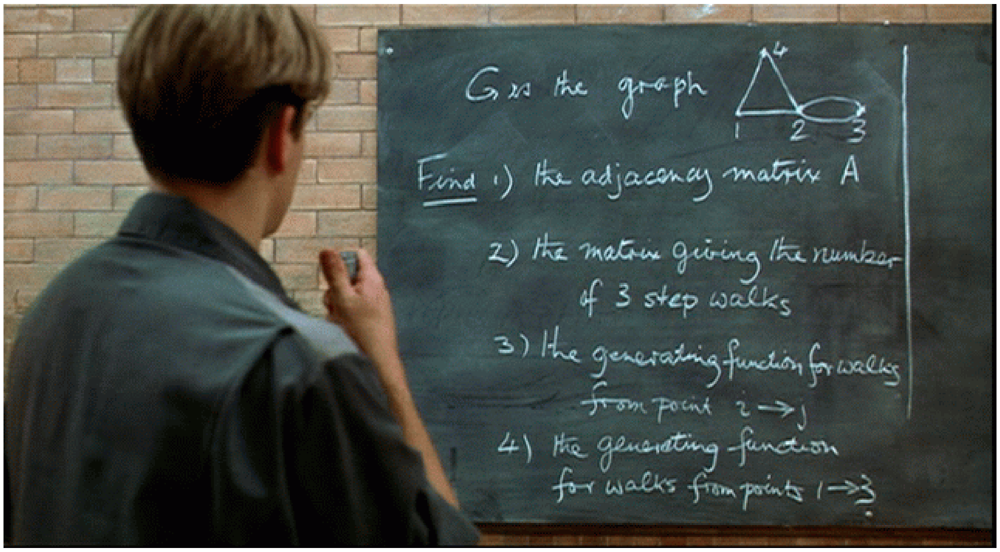

FIGURE 1. The board of the goodwill problem.

Problem 10.4: The link structure of the web forms a graph, where the individual websites are the nodes and if there is an arrow from site a_i to site a_j if a_i links to a_j. The adjacency matrix A of this graph is called the **web graph**. If there are n sites, then the **adjacency matrix** is a $n \times n$ matrix with entries $A_{ij} = 1$ if there exists a link from a_j to a_i. If we divide each column by the number of 1 in that column, we obtain a Markov matrix A which is called the **normalized web matrix**. Define the matrix E which satisfies $E_{ij} = 1/n$ for all i, j. The graduate students and later entrepreneurs **Sergey Brin** and **Lawrence Page** had in 1996 the following "one billion dollar idea": A **Google matrix** is the matrix $G = dA + (1-d)E$, where $0 < d < 1$ is a parameter called **damping factor** and A is the stochastic matrix obtained from the adjacency matrix of a graph by scaling the rows to become stochastic matrices. This is a stochastic $n \times n$ with eigenvalue 1. The corresponding eigenvector v scaled so that the largest value is 10 is called **page rank** of the damping factor d. Page rank is probably the world's largest matrix computation. In 2006, the graph had n=8.1 billion nodes. Find the google matrix in the case of the "claw graph". This is a graph with a central node connected to 3 neighbors which are not connected to each other. Find the page rank in the following situation: Consider 3 sites A, B, C, where A is connected to B, C and B is connected to C and C is connected to A. Find the page rank to $d = 0.1$.

Problem 10.5: Investigate why the random walk on the Bethe lattice B_d (the Cayley graph of the free group F_d with d generators) is recurrent if and only if $d = 1$ ($F_1 = \mathbb{Z}$).