PROBABILITY THEORY

MATH 154

Homework 9

Central Limit

Problem 9.1: a) Verify that the characteristic function of an exponential distributed random variable X with parameter λ is $\phi_X(t) = \lambda/(\lambda - it)$.
b) The Erlang distribution $Erlang(n, \lambda)$ is the distribution of the sum S_n of n exponentially distributed random variables X_k. Use this to see that a random variable with this distribution satisfies $\mathrm{E}[S_n] = n/\lambda$ and $\mathrm{Var}[S_n] = n/\lambda^2$ and that the characteristic function of S_n is $\phi_{S_n} = \lambda^n/(\lambda - it)^n = (1 - it/\lambda)^{-n}$.
c) Verify that $S_n^* = (\lambda S_n - n)/\sqrt{n}$ has the characteristic function $\phi_{S_n^*}(t) = e^{-it\sqrt{n}}(1 - it/\sqrt{n})^{-n}$.
d) Can you see that in this particular case (as the CLT shows in full generality) that $\phi_{S_n^*}(t)$ converges pointwise to $e^{-t^2/2}$?

Problem 9.2: a) Verify that the differential entropy of the Cauchy distribution with density $1/(\pi(1+x^2))$ is $\log(4\pi)$.
b) As a flashback to midterm tell why the Cauchy distribution is a fixed point of the renormalization map $\mathbb{R}(X) = (X+Y)/2$, where Y is a random variable with the same distribution than X that is independent of X.
c) We have seen in class that the central limit theorem can be seen as the statement that $N(0,1)$ is a fixed point of the renormalization map $\mathbb{R}(X) = (X+Y)/\sqrt{2}$. How come, the Cauchy distributed random variable does not get attracted to that fixed point?
d) Verify that the renormalized variance $\lim_{n\to\infty} \frac{1}{n} \int_{-n}^{n} x^2 f(x)\, dx$ exists for the Cauchy distribution. What is its value?

Problem 9.3: a) Compute the entropy of the standard distribution $N(0,1)$. We have sketched it in class.
b) What is bigger, the entropy of the Cauchy distribution or the entropy of the standard normal distribution?
c) We showed in class that the Normal distribution has maximal entropy. Does this not clash with b)?
d) Compute the entropy of the exponential distribution on $[0,\infty)$ with parameter λ. Can it become larger than the normal distribution? Why does this not clash with the fact that the normal distribution has maximal entropy?

Problem 9.4: We work here with measures on $(\mathbb{R},\mathcal{B})$.
a) Assume $d\mu(x) = f(x)\ dx$ and $d\nu(x) = g(x)\ dx$ are absolutely continuous probability measures. The convolution $f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)\ dy$ defines a new measure $d\mu * d\nu = f * g\ dx$. Verify

$$\int f * gh(z)\ dz = \int \int h(x+y)f(y)\ dyg(z)\ dz\ .$$

b) Conclude that

$$d\mu * d\nu(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(x+y)\ d\mu(x)\ d\nu(y)$$

c) Verify that the transformation

$$T_\lambda(\mu)(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_A(\frac{x+y}{\sqrt{2}})\ d\mu(x)\ d\mu(y)$$

on the space of all Borel probability measures on $(\mathbb{R},\mathcal{B})$ satisfying $\int x\ d\mu(x) = 0$ and $\int x^2 d\mu(x) = 1$ has a unique fixed point.

Problem 9.5: We have seen that the central limit theorem implies the de Moivre central limit theorem so that in principle we do not need to prove it again. Write down a proof of the de Moivre central limit theorem. You have the following options: (you only have to do one):
a) using the Stirling approximation formula $n! \sim \sqrt{2\pi n}(n/e)^n$ for the factorial.
b) using characteristic functions, essentially repeating the general proof.