PROBABILITY THEORY

MATH 154

Homework 8

Transformation

Problem 8.1: a) Verify that the **cat map** on the torus $\mathbb{T}^2$ given by the transformation $T(x, y) = (2x + y, x + y) \bmod 1$ defines a measure preserving transformation.
b) How come that if we look at the cat A in a 1000 x 1000 pixel resolution there exists some n such that $T^n(A) = A$. The cat picture gets restored!
c) Modify the proof shown in class for the map $T(x) = 2x$ to see that the cat map is ergodic. Use Fourier series $X(x, y) = \sum_{n,m} a_{n,m} z^n w^m$, where $z = e^{2\pi ix}$ and $w = e^{2\pi iy}$.

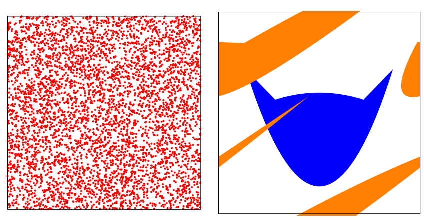

Figure 1. 5000 orbits of a point under the cat map of Arnold. Arnold illustrated how a poor cat on the torus gets distorted under the evolution. We see one iteration.

Problem 8.2: a) Verify that automorphisms of a probability space form a group. Investigate whether (i) ergodic transformations, (ii) weakly mixing transformations or (iii) mixing automorphisms form a subgroup.
b) For every $T \in \mathrm{Aut}(\Omega, \mathcal{A}, \mathrm{P})$ we have a unitary transformation $U : \mathcal{L}^2 \to \mathcal{L}^2$ given by $Uf = f(T)$. Check the orthogonality condition $\langle Uf, Ug \rangle = \langle f, g \rangle$.
c) Classical mechanics is the theory of automorphisms of probability spaces, which leads to unitary maps $Uf = f(T)$. Quantum mechanics deals with the larger automorphism group of **all** unitary operators. Assume our probability space is finite. What is its classical automorphism group? What is its quantum automorphism group?

Ergodicity

Problem 8.3: Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space, and let $T : \Omega \to \Omega$ be a measure-preserving transformation. Verify that the following conditions are equivalent:
(i) T is ergodic
(ii) If $A \in \mathcal{A}$ and $\mathrm{P}[T^{-1}(A)\Delta A] = 0$, then $\mathrm{P}[A] = 0$ or $\mathrm{P}[A] = 1$.
(iii) If $A \in \mathcal{A}$ satisfies $\mathrm{P}[A] > 0$ then $\mathrm{P}[\bigcup_n T^{-n}(A)] = 1$.
(iv) If $A, B \in \mathcal{A}$ satisfy $\mathrm{P}[A] > 0, \mathrm{P}[B] > 0$ then there is n such that $\mathrm{P}[T^{-n}(A) \cap B] > 0$.
Instead of checking all 12 possible ordered pairs, use the Merry-Go-Round proof technique: $(i) \to (ii) \to (iii) \to (iv) \to (i)$.

Weak mixing

Problem 8.4: a) In the proof showing that T is mixing implies T^2 is mixing, we use the following Lemma from calculus or real analysis: the following two things are equivalent for a sequence c_n of numbers $c_n \geq 0$ that are globally bounded: $\exists M > 0$ such that $0 \leq c_n \leq M, \forall n \geq 0$.
(i) $\frac{1}{n}\sum_{k=1}^{n} c_k \to 0$.
(ii) There exists a set J of density 1 in $\mathbb{N}$ on which $\lim_{j \in J} c_k \to 0$.
b) Use a) to show $\frac{1}{n}\sum_{k=1}^{n} |c_k| \to 0$ is equivalent to $\frac{1}{n}\sum_{k-1}^{n} |c_k|^2 \to 0$.
c) Conclude that weakly mixing can be rephrased as the property $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\mathrm{P}[A \cap T^{-k}(B)] - \mathrm{P}[A]\mathrm{P}[B]|^2 = 0$ for all $A, B \in \mathcal{A}$.

Mixing

Problem 8.5: a) Prove the following result of Rényi: A dynamical system T is mixing if and only if $\mu(A \cap T^{-n}A) \to \mu(A)^2$ for $n \to \infty$.
b) State and give a proof of the Riemann-Lebesgue lemma. Why does this lemma imply that T has only absolutely continuous spectrum, then T is mixing? (Use a).

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