

PROBABILITY THEORY

MATH 154

Homework 6

STOCHASTIC CONVERGENCE

- Problem 6.1:**
- a) Is there for $1 \leq p < \infty$ a relation between L^p convergence and convergence almost everywhere?
 - b) Is there a relation between L^∞ convergence and convergence almost everywhere?
 - c) Is there a relation between complete convergence and L^p convergence for $p < \infty$?
 - d) Is there a relation between complete convergence and L^∞ convergence?

Problem 6.2: Consider the random variables $X_n(x) = \cos(nx)$ on $[-\pi, \pi], \mathcal{B}, dx/(2\pi)$.

- a) By writing $\cos(nx) = \operatorname{Re}(e^{inx})$ and using a geometric series, verify that

$$S_n(x) = \frac{2 \sin((n+1/2)x)}{\sin(x/2)} - 1.$$

This is $D_n(x) - 1$, where $D_n(x)$ is called the Dirichlet kernel.

- b) Verify that $\|S_n(x)\|_1 + 1 \geq \frac{2}{\pi} \log(2n+1)$.

A proof can be found on the Wikipedia page on the Dirichlet kernel. Write it down in your words.

- c) First recollect from class why the assumptions of the weak law are satisfied and restate the conclusion of that theorem about S_n/n . This should verify that $S_n/n \rightarrow 0$ in $\mathcal{L}^1$ and so in probability.
- d) Given a continuous even function f with $E[f] = 0$, the expectation $a_n = E[fX_n]$ is called the n 'th Fourier coefficient and $g(x) = \sum_n a_n X_n(x)$ is the cos-Fourier series of n . The formula $\sum_{n=1}^{\infty} a_n^2 = \|f\|_2^2$ is called Parseval's identity. What geometric condition does assure it and what famous geometric theorem does it generalize?

Problem 6.3: a) Give an example of a sequence of random variables $X_n \rightarrow X$ for which we have convergence in probability but not complete convergence.
 b) Give an example of a sequence of random variables X_n , where $X_n \rightarrow X$ in probability but where $X_n \rightarrow X$ in $\mathcal{L}^1$ does not happen.
 c) Assume $X_n \rightarrow X$ in $\mathcal{L}^3$? For which p can one say $X_n \rightarrow X$ in $\mathcal{L}^p$?
 d) Give an example of a sequence of random variables where $X_n \rightarrow X$ in $\mathcal{L}^1$ but where the convergence is not in $\mathcal{L}^2$.

LAW OF LARGE NUMBERS

Problem 6.4: The n 'th Chebyshev polynomial is defined as $X_n = T_n(x) = \cos(n \arccos(x))$. We have $T_n(\cos(t)) = \cos(nt)$.
 a) Verify that $T_n(x)$ is a polynomial of degree n and write down $T_n(x)$ for $n = 1, 2, 3, 4$.
 b) We look at $T_n(x)$ as a random variable on the probability space $(\Omega = [-1, 1], \mathcal{B}, P = \frac{1}{\pi\sqrt{1-x^2}})$. Check that the latter indeed is a probability space.
 c) Demonstrate (by showing all conditions) that you can use the weak law of large numbers to establish that $(1/n)S_n$ converges in probability to 0.

Problem 6.5: Let $X_n(\omega)$ be the n 'th binary digit of $\omega \in [0, 1]$.
 a) Investigate the convergence $\frac{S_n}{n} \rightarrow m$ in probability.
 b) Verify that S_n has the Binomial distribution $p_k = \binom{n}{k} 1/2^n$.
 c) Show directly and then use the weak law to see that $S_n/n \rightarrow 0$ in probability.
 d) Verify that $S_n/\sqrt{n}$ does not go to zero in L^2 .
 e) Verify also that $S_n/\sqrt{n}$ does not converge to 0 in distribution.