PROBABILITY THEORY

MATH 154

Homework 2

Probability spaces

The axioms of a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ are:
1) $\mathcal{A}$ is a **σ-algebra**: it is a Boolean ring on which any set operation can be done finitely or countably many times:

(i) $(\mathcal{A}, +, \cdot, 0 = \emptyset, 1 = \Omega)$ is a Boolean ring.
(ii) $A_n \in \mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$.

As we have seen in class, you can assume that all other set operations are allowed, like $A^c = \Omega \setminus A = 1 + A$, $A \cup B = (A^c \cap B^c)^c$ or $A \cup B = A + B + AB$.

2) P is a **probability measure**: it is non-negative, normalized and σ-additive.

(i) $\mathrm{P}[A] \geq 0$
(ii) $\mathrm{P}[\Omega] = 1$
(iii) $A_n \in \mathcal{A}$ disjoint $\Rightarrow \mathrm{P}[\bigcup_n A_n] = \sum_n \mathrm{P}[A_n]$

Problem 2.1: Verify the following properties from the axioms:
a) $\mathrm{P}[\emptyset] = 0$.
b) $A \subset B \Rightarrow \mathrm{P}[A] \leq \mathrm{P}[B]$.
c) $\mathrm{P}[\bigcup_n A_n] \leq \sum_n \mathrm{P}[A_n]$.
d) $\mathrm{P}[A^c] = 1 - \mathrm{P}[A]$.
e) $0 \leq \mathrm{P}[A] \leq 1$.
f) $A_1 \subset A_2, \subset \cdots$ with $A_n \in \mathcal{A}$ then $\mathrm{P}[\bigcup_{n=1}^\infty A_n] = \lim_{n\to\infty} \mathrm{P}[A_n]$.

Problem 2.2: Let Ω be a set. Let $\mathcal{A}$ be the set of countable or co-countable subsets of Ω.
a) Verify that $\mathcal{A}$ satisfies all the ring axioms of a Boolean algebra.
b) Verify that $\mathcal{A}$ is a π-system.
c) Verify that $\mathcal{A}$ is a λ-system.
d) Verify that $\mathcal{A}$ is a σ algebra without using the theorem of Lecture 3.
e) Verify that $\mathcal{A}$ is the smallest σ-algebra containing the cofinite topology.

Problem 2.3: Let $\Omega = [0,1]^2$. Let $\mathcal{I} = \{[a,b) \times [c,d)\}$ denote the set of all left-bottom closed right-top open rectangles.
a) Verify that this is a π-system.
b) Verify that $\mathrm{P}[a,b) \times [c,d)] = (d-c)(b-a)$ is a probability measure on this π system.
c) Why can the measure P be extended to the smallest σ-algebra containing $\mathcal{I}$?
d) Under which conditions are two elements in $\mathcal{I}$ independent?

Problem 2.4: Verify the following properties. The first four are known as **Keynes postulates**, the fifth is called **Bayes Theorem**.
1) $\mathrm{P}[A|B] \geq 0$.
2) $\mathrm{P}[A|A] = 1$.
3) $\mathrm{P}[A|B] + \mathrm{P}[A^c|B] = 1$.
4) $\mathrm{P}[A \cap B|C] = \mathrm{P}[A|C] \cdot \mathrm{P}[B|A \cap C]$.
5) $\mathrm{P}[A|B| = \mathrm{P}[B|A]\mathrm{P}[A]/\mathrm{P}[B]$.

Problem 2.5: Prove the $\Pi\Sigma\Lambda$ **sorority theorem** in the text. It states "The smallest λ-system $\mathcal{A}$ containing a π-system $\mathcal{I}$ is the smallest σ algebra containing $\mathcal{I}$."

FIGURE 1. To the left an example of a $\Pi\Sigma\Lambda$ chapter (in this case Oxford MS). To the right, a brooch from BU in the shape of a Marguerite daisy (or $A \cap B$ when intersecting two sets in a Venn Diagram) also in the order of the mathematical order $\Pi\Lambda\Sigma$: to check that we have a σ-algebra, we have to check it is a π-system and a λ-system.

Oliver Knill, knill@math.harvard.edu, Math 154, Spring, 2026