CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 30: Mass Spring Systems

Hooks law

30.1. If $x(t)$ is the position of a mass point of mass m, **Newton's law** relates mass times acceleration $mx''(t) = F(x(t), x'(t))$ with the force F. In the simplest case, when $F(x) = -cx$, this is the **Hook law** describing the **frictionless mass-spring system** $x'' = -cx$ with $c = f/m$, where f is the spring constant and m is the mass. In that case $C_1 \cos(\sqrt{c}t) + C_2 \sin(\sqrt{c}t)$ are solutions as one can check by differentiating twice. With friction, the law generalizes to $x'' = -bx' - cx$, where b is the **friction coefficient**. How do we solve this?

Characteristic equation

30.2. Here is an example of such a system. How do we solve

$$x'' + 7x' + 10x = 0 \ ?$$

The trick is to input $x(t) = e^{rt}$. One calls this also "Ansatz" which is German for "lets try!" We get

$$r^2 e^{rt} + 7re^{rt} + 10e^{rt} = 0 \ .$$

and because e^{rt} is never zero, we can divide it through to get the **characteristic equation**

$$r^2 + 7r + 10 = 0$$

It has the roots $r = -5$ and $r = -2$. Therefore, the general solution is

$$x(t) = C_1 e^{-5t} + C_2 e^{-2t} \ .$$

If we know the initial position $x(0) = 2, x'(0) = -7$ we can fix the constants $C_1 = 1$ and $C_2 = 1$.

Complex solutions

30.3. For the friction-less system $x'' = -x$, the characteristic equation is

$$r^2 + 1 = 0 \ .$$

The only roots are **complex** $\pm\sqrt{-1} = \pm i$ leading to the general solution $C_1 e^{it}$ and $C_2 e^{-it}$. We also know already that $C_1 \cos(t) + C_2 \sin(t)$ is a solution. A relation of Euler links the two. It is not part of the courses this semester. Since all surveys without exception give as the most beautiful equation in mathematics the winner:

$0 = 1 + e^{i\pi}$.

It combines **geometry** π, **analysis** e, and **algebra** i. Arithmetic comes in with **addition** with neutral 0 and **multiplication** with neutral 1 and **exponentiation**. This formula is a special case of the **Euler formula**

$e^{it} = \cos(t) + i\sin(t)$.

Just plug in $t = \pi$ to get $e^{i\pi} = -1$. This is the winner of all math beauty contests.

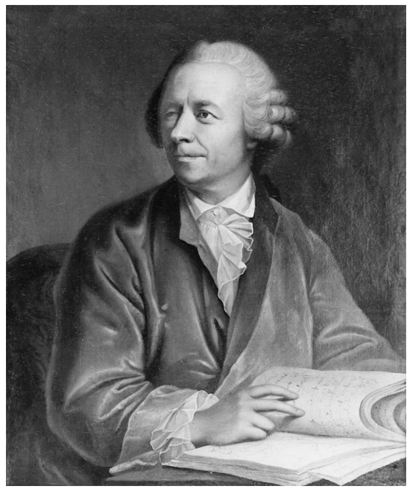

FIGURE 1. Leonhard Euler (1707-1783)

30.4. The Euler formula can be proven by **Taylor series**. Remember $e^x = 1 + x + x^2/2 + x^3/6 + ...$ and $\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$ and $\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$. Now plug in $x = it$ into the exponential function to get $e^{it} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} \cdots$. You see that the real part of the later is just $\cos(t)$ and that the imaginary part is $\sin(t)$.

THE QUADRATIC EQUATION

30.5. Complex numbers appear when solving the **quadratic equation**

$$r^2 + br + c = 0 \ .$$

Rewriting it as $r^2 + br + b^2/4 = b^2/4 - c$ allows to see the left is a square $(r + b/2)^2 = b^2/4 - c$. Multiply both sides with 4 and take the square root to get $2(r + b/2) = \sqrt{b^2 - 4c}$. Solving for r gives:

$r = \frac{-b\pm\sqrt{b^2-4c}}{2}$ solves $r^2 + br + c = 0$.

30.6. We see that every quadratic polynomial has exactly two roots! They are complex in general but they exist. If $b^2 - 4c$ is negative, we get complex solutions r_1, r_2. The solution of the above spring system $x'' + bx + c$ is now $x(t) = C_1e^{r_1t} + C_2e^{r_2t}$.