CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 29: Mixing Problems

Input Output system

29.1. Autonomous input-output system $y' = a + ry$ occur common in mathematics. An example is a **banking problem** where a is a constant income and r is an interest rate. It also occurs in other **input-output problems** for concentrations, where a constant amount is entering with a given concentration and a part depending on y is leaving. In the later case, it can happen that r can depend on t. The difficulty of these problems is to translate a given word problem into the differential equation.

> **Example:** *A 20-liter juice dispenser in a cafeteria is filled with juice mixture that is 10 percent mango juice and 90 percent cranberry juice. An orange-mango blend that is 50 percent orange and 50 percent mango is entering the dispenser at a rate of 4 liters per hour and the well-stirred mixture leaves at the same rate. Find the differential equation for the Mango concentration $M(t)$.*

To solve this, first list the set-up in an organized way.

Input: 2 Liters per minute Mango and 2 Liters per minute Orange are entering.

Container: 20 Liter tank filled with 2 liters Mango and 18 liters of Cranberry.

Output: 4 Liters per minute are leaving the dispenser.

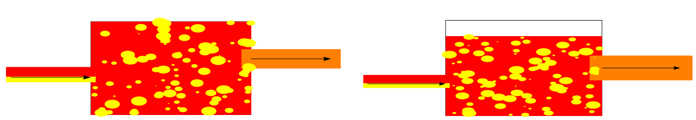

Figure 1. To the left, a juice machine where an equal amount of liquid leaves than what enters. To the right, more liquid leaves than enters.

29.2. To set up a differential equation, we write down the rate of change of $M(t)$. It is a difference between input and output. The input is constant 2. The output is 4 times the current Mango concentration $M(t)/20$, which is $M(t)/5$. We have

$$M'(t) = 2 - 4M(t)/20 = 2 - M(t)/5$$

and $M(0) = 2$. The $\boxed{M(t) = 10 - e^{-t/5}}$.

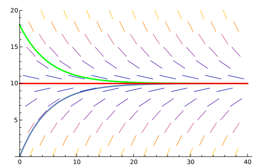

FIGURE 2. The slope field picture with trajectories of the Mango-Cranberry-Orange system. The value $M = 10$ is the only equilibrium.

29.3. Now modify the situation where the amount of juice leaving is 6 liters per minute. This is larger than the amount coming in.
$\boxed{\text{Input}}$: 2 Liters per minute Mango and 2 Liters per minute Orange are entering.
$\boxed{\text{Container}}$: 20 Liter tank filled with 2 liters Mango and 18 liters of Cranberry.
$\boxed{\text{Output}}$: 6 Liters per minute are leaving the dispenser.
In this new situation, the tank volume will decreases like $V(t) = 20 - 2t$. The Mango concentration will be $M(t)/(20-2t)$. With 6 liters leaving, the amount of Mango juice leaving is $6M(t)/(20 - 2t)$. We get the differential equation

$$M'(t) = 2 - \frac{6M(t)}{20 - 2t} .$$

The differential equation can still be solved. The solution is

$$M(t) = 10 - t + (t - 10)^3/100$$

Of course, at time $t = 10$, there will be no juice any more in the tank and the model stops to make sense.

30. Intro to second order systems

30.1. In this lecture, we also start to look at second order differential equations like

$$y''(t) = f(t) .$$

This is important in physics because $y''(t)$ is the **acceleration**. For example,

$$y''(t) = -10$$

is a **free fall problem**. We can get $y(t)$ by integration, in the same way as we did for first order systems. However, we have to integrate twice. This implies that we have to fix two constants. This makes sense. If we throw an object, we must know both the initial velocity and initial position. The general solution of $y(t)$ is

$$-5t^2 + y'(0)t + y(0) .$$

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