CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 28: Separable systems

Separable systems

28.1. A differential equation that can be written as $dy/dt = f(t)g(y)$ is called **separable**. The reason for the name is that we can move all terms containing t onto the left and all terms containing y to the right, integrate and then get the solution.

How does it work?

28.2. Lets look at the example $\frac{dy}{dt} = -\frac{t}{y}$ seen earlier in the course. Write

$$ydy = -tdt$$

then integrate

$$\int ydy = \int -tdt + C$$

to get $y^2 = -t^2 + C$ or equivalently $y^2 + t^2 = C$. The solution curves $y(t)$ are located on circles. The line $t = 0$ consists of equilibria.

Why does it work?

28.3. The above calculation is elegant and efficient. The justification why we can do the magic requires a change of variables. Write

$$yy'(t) = -t$$

and now integrate both sides with respect to t.

$$\int y(t)y'(t)\ dt = -\int t\ dt\ .$$

Using the chain rule (or reverse substitution), we get

$$\int ydy = -\int t\ dt$$

which is the expression we got the fast way above.

Lets practice

28.4. Problem: Solve $y' = (1 + y^2)\sin(t)$.
Solution: $dy/(1 + y^2) = \sin(t)dt$ so that $\arctan(y) = -\cos(t) + C$ and so $y(t) = \tan(-\cos(t) + C)$.

28.5. Problem: Solve $y' = (t^2 + t)/y^9$.
Solution $dyy^2 = (t^2+t)dt$ so that $y^3/3 = t^3/3+t^2/2+C$ and $y(t) = (t^3+3t^2/2+3C)^{1/3}$.

28.6. Problem: Solve $y' = ry-a$. We have $dy/(ry-a) = dt$ so that $\ln(ry-a)/r = t+C$ so that $ry - a = e^{rt+rC}$ which means $y(t) = a/r + ce^{rt}$ where $c = e^{rC}/r$ is an other constant. We see that the inhomogeneous equation is the sum of the equilibrium a/r and the general solution ce^{rt} of the equation $y' = ry$.

Does it always work?

28.7. As usual with integration, one has to be able to solve the integrals in order to get explicit solutions. If either $1/f(t)$ or g do not have anti derivatives, we can not get there.

28.8. An other difficulty which can occur is if the differential equation does not have a unique solution like

$$y' = \sqrt{y}$$

In that case the integration gives $\int dy/\sqrt{y} = \int dt = t + C$ so that $2\sqrt{y} = t + C$ and so $y = (t + C)^2/4$. If $y(0) = 0$, then $y(t) = t^2/4$ is a solution. But also $y(t) = 0$ is a solution.

Who invented it?

28.9. The name l'Hospital is usually used for the invention. Note however that l'Hospital was working for Johann Bernoulli and Bernoulli claimed that several ideas attributed to Hospital were actually his own. See C. Truesell: The New Bernoulli Edition, Isis Vol 49 No 1, 1958, pages 64-62.