CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 24: Overview

Summary

Find a finite Taylor polynomial $P_n(x)$ or Taylor series. Relate coefficients a_k with the derivative $f^{(k)}(c)$ in case of a Taylor series. Understand what convergence $s_n \to s$ means. Know how to check **convergence or divergence** for **geometric series**, **alternating series**, **p-series**, **Power series** and **Taylor series**. Find **interval of convergence** including behavior on the end points. Use **power series** to find new series like $x^8 e^{-x^7}$ or integrals. Apply convergence tests: **n-term test**, **asymptotic comparison**, **direct comparison**, **integral comparison**, **ratio test**, **alternating series test**, **absolute convergence test**, **monotone convergence test**. Estimate **remainder errors** in case of alternating series, geometric series or Taylor series. Especially the **remainder theorem** for Taylor series. Understand and draw **geometric pictures** like the **partial sum diagram** or the **function picture** when comparing with integrals.

Series

24.1. Some notation.

Sum or series	$s = \sum_{k=0}^{\infty} a_k$
Partial sum	$s_n = \sum_{k=0}^{n} a_k$
Error term	$s - s_n = \sum_{k=n+1}^{\infty} a_k$
Taylor polynomial	partial sum to **Taylor series**
Polynomial	partial sum to **power series**

Convergence

24.2. We distinguish:

Convergence	there is a finite s such that $s_n \to s$
Divergence	there is no finite S such that $s_n \to s$
Absolute convergence	$\sum_{k=0}^{\infty} \lvert a_k \rvert$ converges
Conditional convergence	convergence but not absolute
Maximal open Interval of convergence	$(c-R, c+R)$
Interval of convergence	could include also some boundary point(s)
Radius of convergence	radius R of open interval

Convergence tests

24.3. The tests we have seen are:

N-term test	$a_k \nrightarrow 0$ implies divergence
Alternating series test	a_k alternating $\lvert a_k\rvert \to 0$ monotonically
Ratio test	$\lvert a_k/a_{k+1}\rvert \to r < 1$
Direct Comparison test	$0 \leq \lvert a_k\rvert \leq b_k$ and $\sum_k b_k$ converges
Asymptotic Comparison test	$a_k \sim b_k$ and $\sum_k b_k$ converges
Integral comparison test	$\lvert a_k\rvert \leq f(k)$ and $\int_n^\infty f(x)\, dx$ converges
Monotone convergence test	$a_k \geq 0$ and s_n bounded above gives convergence

Types of series

24.4. Be friends with the following series:

Finite geometric sum	$\sum_{k=0}^{n-1} ar^k = a(1-r^n)/(1-r)$
Geometric series	$\sum_{k=0}^{\infty} kar^k = a/(1-r)$
Power series	$\sum_{k=0}^{\infty} a_k(x-c)^k$
Taylor series	$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$
Alternating series	$\sum_{k=0}^{\infty}(-1)^k b_k$ with b_k not changing sign.

Error bounds

24.5. Know your errors:

Taylor series	$M_{n+1}\frac{\lvert x-c\rvert^{n+1}}{(n+1)!}$
Alternating series	a_{n+1}
Geometric series	$\frac{a_{n+1}}{1-r}$
Integral approximation	$\int_n^\infty f(x)\, dx$

Hall of fame

24.6. These are the super stars of series:

Geometric series	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$	2
Harmonic series	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...$	∞
Basel series	$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+...$	$\pi^2/6$
Euler series	$1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$	e
Log(2) series	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...$	log(2)
Leibniz series	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...$	$\pi/4$
Grandi series	$1-1+1-1+1-...$	1/2 (*)
Silly series	$1+1+1+1+1+...$	$-1/2$ (*)