CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 21: Power series

Power series

21.1. A series

$$S(x) = \sum_{k=0}^{\infty} a_k x^k \; .$$

is a **power series**. More generally one can center them at a point c

$$S(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \; .$$

21.2. An important subclass of power series are **Taylor series**

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \; .$$

Example: a) $\sum_{k=0}^{\infty} x^k$ which is the Taylor series of $1/(1-x)$.
b) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ which is the Taylor series of $\exp(x)$.
c) The power series $\sum_k k! x^k$ is not a Taylor series.

Radius of convergence

21.3. For any power series, there is an **maximal open interval of convergence** $(c-R, c+R)$ centered at c on which the series converges. (The interval of convergence could also include zero, one or two of the boundary points). The number R is called the **radius of convergence**. $R = 0$ happens for $\sum_k k! x^k$, the case $R = \infty$ happens for $\sum_k x^k/k!$ and means the interval is the $\mathbb{R}$.

21.4. If R is the radius of convergence then for $|x-c| < R$, the series converges. For $|x-c| > R$ the series is divergent.

Example: For $\sum_{k=1}^{\infty} \frac{x^k}{k}$ for example, the series converges for $|x| < 1$ by the **ratio test**, we have seen before. For $|x| > 1$ the series diverges by the n'th term test.

For $x = 1$, we have the **Harmonic series**, where the series diverges. For $x = -1$ we have an **alternating series** which converges. The limit is $\log(2)$. We say then the"maximal interval of convergence" is $[-1, 1)$ as the left point is included.

21.5. Here is a formula for the radius:

$R = \lim_{k\to\infty} |a_k|/|a_{k+1}|$, if the limit exists.

21.6. Indeed, for $|x-c| < R$, the terms $b_k = a_k(x-c)^k$ satisfy $|b_{k+1}|/|b_k| = |x-c|/R < 1$ so that $\sum_k b_k$ converges by the **ratio test**. For $|x-c| > R$, the terms satisfy $|b_{k+1}|/|b_k| = |x-c|/R > 1$ so that $\sum_k b_k$ diverges by the **n'th term test**.

Why are they useful?

21.7. Power series are mostly used for **Taylor series**. This allows us to work with functions by taking **polynomial approximations** with control about the error. They are also used in combinatorics as **formal power series** where we do not care about convergence.

21.8. A power series not coming from a named function f is the **prime function**

$$S(x) = \sum_{p \text{ prime}} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + \cdots .$$

The coefficients of $S(x)^2 = x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + \dots$ tell in how many ways one can write a given number as a sum of two primes. The **Goldbach conjecture** states that every even number can be written as such. In other words, all even derivatives of $S(x)^2$ should be non-zero. The **Goldbach comet** plotting the coefficients of the power series of $S(x)^2$ illustrates that.

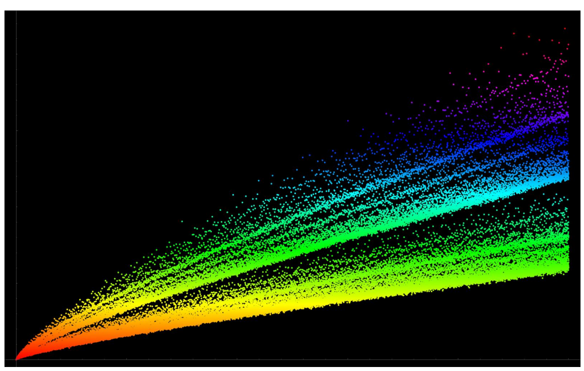

FIGURE 1. The Goldbach comet gives the coefficients of b_k.

21.9. An other application are **moment generating function** in **statistics**.

$$S(x) = \sum_{n=0}^{\infty} \frac{E[X^n]x^n}{n!}$$

where $E[X]$ denotes the expectation or mean of a random variable X. Moments are important because for example the variance of a random variable is $E[X]^2 - E[X]^2$ and so expressible using moments.