CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 20: Ratio Test

Geometric Series review

20.1. Remember that a series with $a_{k+1} = ra_k$ is a **geometric series**. An example is

$$S = 1/8 + 1/16 + 1/32 + 1/64 + \ldots .$$

If a is the first term then the sum S is $a/(1-r)$. One reason is that the function $f(x) = a/(1-x)$ has a Taylor series $a + ax + ax^2 + ax^3 + \ldots$. Putting $x = r$ gives the result.

Ratio test

20.2. We do not actually need to have a perfect geometric series to decide about convergence. The **ratio test is**

If $|a_{k+1}|/|a_k| \to r < 1$ then $\sum_k |a_k|$ converges.

20.3. The reason is as follows. Let s be a number larger than r but smaller than 1. The assumption of the test implies that $|a_{k+1}|/|a_k| \leq s$ for large enough k. This assures that $|a_k| \leq Cs^k = b_k$ for some constant C. Now use the **direct comparison test** to see that $\sum_k |a_k|$ converges.

20.4. The ratio test was first formulated by Jean Le Rond d'Alembert. It appears in the work "Opuscules" published in 1768.

Examples

20.5.

Example: Use the ratio test in the case $\sum_k a_k = \sum_k 1/1.1^k$. We have $a_{k+1}/a_k = 1.1^k/1.1^{k+1} = 1/1.1 < 1$. The ratio test assures that we have convergence. Indeed, we had a geometric series with $r = 1/1.1$.

20.6.

Example: Does the sum $\sum_k a_k = \sum_k k/2^k$ converge?
Answer: Yes. We use the ratio test. We have $a_{k+1}/a_k = (k+1)/(2k)$. Using l'Hosptial, we see that the limit is $1/2$. As this is smaller than 1 we have convergence.

Figure 1. Jean Le Rond d'Alembert, 1717-1883

20.7.

Example: Use the ratio test to show that $\sum_k 2^k/k!$ converges. **Answer:** . Use the ratio test. We have $a_{k+1}/a_k = 2/(k+1)$. As this converges to 0, we have convergence. Actually, we see that we can replace 2^k with any x^k and still have convergence.

20.8.

Example: What does the ratio test tell in the case $a_k = 1/k$? We have $a_{k+1}/a_k \to 1$. This is not sufficient. The ratio test does not apply. Indeed, the series diverges.

20.9.

Example: What does the ratio test tell in the case $a_k = 1/k^2$? Answer: We have $a_{k+1}/a_k = k^2/(k^2+2k+1)$ also here, the limit is 1 and the ratio test does not apply. But we know that the sum converges as it is the $p = 2$-series. The ratio test does not allow us to forget about p-series!

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