CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 16: Geometric series

Geometric series

16.1. The **geometric series** $S = \sum_{k=0}^{\infty} x^j$ is no doubt the most important series in mathematics. Do not mix it up with $S = \sum_{j=1}^{\infty} k^x$ which is called the **zeta function** which is written as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. It is custom to write the geometric series as $\sum_{n=0}^{\infty} ar^n$ so that a is the first term and the term following the next is r times that number.

16.2. a) A first justification for

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

is to multiply both sides with $1 - r$.
b) A second justification is to make the Taylor expansion of $f(x) = \frac{1}{1-x}$ at $x = 0$ and to see that $f^{(k)}(0) = k!$. c) A third justification is to look at the **finite sum** formula for the partial sum S_n below and which generalizes expressions like $1 + x = \frac{x^2-1}{x-1}$ or $1 + x + x^2 = \frac{x^3-1}{x-1}$ or $1 + x + x^2 + x^3 = \frac{x^4-1}{x-1}$.

Convergence

16.3.

The geometric series converges for $|r| < 1$.

The reason is the formula for the **partial sums**

$$S_n = a(1 + r + r^2 + \cdots + r^{n-1}) = a\frac{1-r^n}{1-r} \,.$$

If $|r| < 1$, then $r^n \to 0$ and the partial sums $S_n = a\frac{1-r^n}{1-r}$ converge to $\frac{a}{1-r}$.

16.4. The geometric series diverges for $r = 1$ and $r = -1$. In the case $x = 1$, the partial sums converge to infinity, in the case $r = -1$, we have the Grandi series that does not converge by the n'th term test.

Applications

16.5. Historically, the geometric series first appeared in **Zeno's paradox**. Assume Archilles races a turtle who is given an advance of $a = 1$ miles Assume the turtle can run r times slower than Archilles. Both start running. Once Archilles reaches the start point of the turtle, the turtle is x miles ahead at $S_1 = 1 + r$. Once Archilles the point S_1 the turtle is at $S_2 = 1 + r + r^2$, if Archilles reaches S_2 the turtle is ahead again at $1 + r + r^2 + r^3$. Since the turtle is always ahead of Archilles, Artchilles will never be able to catch up. We see that the fallacy is that an infinite sum is not necessarily infinite. There is a definite point when Archilles will overtake the turtle and that is if Archilles has run $1/(1 - r)$ miles. This works. If both would run with the same speed $x = 1$, then the time to catch up would be infinite.

16.6. Next in history comes a computation of Archimedes about the **quadrature of the parabola**. Look at a line segment connecting $A = (a, a^2), B = (b, b^2)$ in a parabola. This defines a triangle with third point $C = ((a + b)/2, (a + b)^2/4)$. The two new segments AC and BC again define triangles as such of area $x = 1/4$ times of the original triangle. Repeat like this and add up all these triangle areas. They fill up the area bound by the parabola and the original segment. This area is therefore $1/(1 - x) = 4/3$ times the area of the original triangle.

16.7. A nice application is in probability theory. If you do an experiment which succeeds with probability p then the probability to be successful the first time after k steps is $(1 - p)^{k-1}p$. You see $a = p$ and $r = 1 - p$. The total sum is $a/(1 - r) = p/(1 - (1 - p)) = 1$. This probability distribution is called, surprise, surprise, the **geometric distribution**.

16.8. An application appears in fractal geometry. We can compute the **area of the Koch snowflake**.

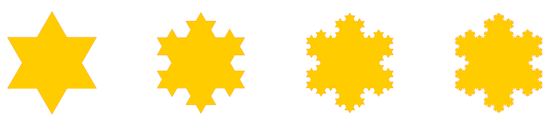

Figure 1. The first 4 approximations of the Koch snowflake.

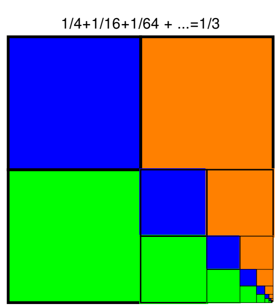

Figure 2. The identity $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}$.