CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 14: Convergence

Convergence of series

14.1.

Definition: A series $\sum_{k=1}^{\infty} a_k$ **converges** if there exists a number S such that the **partial sum** $S_n = \sum_{k=1}^{n} a_k$ has the limit S.

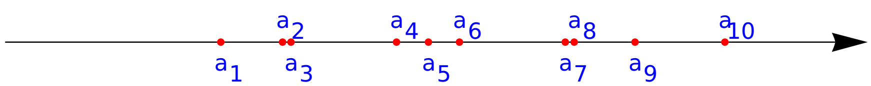

Diagram of partial sums: we took $a_k = \sin^2(2k)$.

Example: The sum

$$S = 1 + 1 + 1 + \dots$$

does not converge. It diverges to infinity because the partial sum is $S_n = n$.

Example: The sum

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots$$

converges to 2. It is called a **geometric series**. It is a **power series** $\sum_{k=1}^{\infty} x^k$ for $x = 1/2$.

Example: The sum

$$S = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots$$

converges. Do you see the limit?

Example: The **Grandi's series** (named after Guido Grandi)

$$S = 1 - 1 + 1 - 1 + 1 \dots$$

also does not converge because the partial sums fluctuate between 0 and 1. We need that $a_n \to 0$.

Example: The **Harmonic series**

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} ...$$

diverges. We have seen it in a homework when doing the Taylor expansion of $\log(1 + x)$ and evaluated at $x = -1$.

Example: The **Leibniz sum**

$$S = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} ...$$

converges and has the value $\pi/4$ as one can see when making the Taylor expansion of $\arctan(x)$ and evaluated this at $x = 1$. The **alternating harmonic sum**

$$S = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} ...$$

converges to $\ln(2)$.

Example: A **Taylor series** $\sum_{n=0}^{\infty} f^{(k}(0)x^k/k!$ converges if the rest term $R_n = M_{(n)}|x|^n/n!$ converges to zero.

Example: Given a constant p, the sum

$$S = 1^{-p} + 2^{-p} + 3^{-p} ...$$

is called a p**-series**. We will see later under which conditions it converges. For $p = 0$, we get the series $f(x) = 1 + 1 + 1....$ seen before

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} ...$$

is an other notation where p is replaced by s and the ζ letter is used. This is the famous **Riemann Zeta function**.

Example: For $s = 2$, we have the so called the **Basel problem**

$$S = 1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ...$$

It is $\pi^2/6$.

We are not yet at **Halloween** but Mathematica tells

$$\zeta(0) = 1 + 1 + 1 + ... = -1/2 .$$

We can evaluate $\zeta(-1)$ which is the sum

$$\zeta(-1) = 1 + 2 + 3 + 4 + 5 + ... = -1/12$$

The Grandi series can be given a Halloween value 1/2 when using the formula $1/(1-x)$ for the sum $\sum_n x^k$ and plugging in $x = (-1)$.

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