CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 9: Improper Integrals

Improper integrals

9.1. When integrating over an infinite interval, or integrating an unbounded function, we get an **improper integral**. We look first at integrals $\int_a^\infty f(x)\,dx$, in the case when f is bounded and continuous.

Definition: If the limit $\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists, we say the integral $\int_a^\infty f(x)\,dx$ **converges**. Otherwise, it **diverges**.

Example: To decide about whether $\int_1^\infty \frac{1}{x^4}\,dx$ converges, integrate from $x=1$ to $x=b$

$$\frac{-1}{3x^3}\Big|_1^b = \frac{1}{3} - \frac{1}{3b^3}\ .$$

and take the limit $b\to\infty$. We see that limit is $1/3$. The integral converges.

Example: What about $\int_1^\infty \frac{1}{x^{1/4}}\,dx$? Since the anti-derivative is $\frac{4}{3}x^{3/4}$, we have $\int_1^b \frac{1}{x^{1/4}}\,dx = \frac{4}{3}x^{3/4}|_1^b = \frac{4}{3}(b^{3/4}-1)$. The limit $b\to\infty$ does not exist. The integral diverges.

Example: The integral $\int_1^\infty \frac{1}{x}\,dx$ diverges because $\int_1^b \frac{1}{x}\,dx = \ln(b) - \ln(1) = \ln(b)$ diverges for $b\to\infty$.

p-integrals

9.2. Bu explicit integration of the **p-integral** $\int_1^b \frac{1}{x^p}\,dx$, (do it yourself!), we see:

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p\le 1$.

9.3. The value $p=1$ is a threshold. It is important that right on the watershed $p=1$, we still have divergence.

Comparison test

9.4. The **comparison test** is

If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty f(x)\, dx$ converges then $\int_a^\infty g(x)\, dx$ converges.

9.5. By reversing the argument, we see

If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\, dx$ diverges, then $\int_a^\infty f(x)\, dx$ diverges.

Example: $\int_1^\infty (5+\sin(x^5))e^{-x}\, dx$ converges because the integrand $g(x)$ is bounded above by $f(x) = 6e^{-x}$, leading to a finite integral.

Example: The integral $\int_1^\infty e^{-3x} \sin(x^2)\, dx$ converges.

Example: The integral $\int_1^\infty \frac{1}{x\ln(x)} + \frac{1}{x^2}\, dx$ diverges.

Example: The integral $\int_0^\infty \sin(x)\, dx$ diverges.

9.6. Note that in the comparison test f, g are assumed to be non-negative. Without that assumption, the result is wrong in general. Can you see why? When dealing with general functions, just take absolute values.

An application

9.7. If the graph of $f(x) = 1/x$ is rotated around the x-axes over the interval $[1, \infty)$, we get an infinite surface called **Gabriel's trumpet**. Its cross section area is π/x^2. The volume up to height b is

$$\int_1^b \frac{\pi}{x^2}\, dx = -\frac{\pi}{x}|_1^b = -\pi/b + \pi \to \pi \ .$$

The **surface area** A of the trumpet is

$$A = 2\pi \int_1^b \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx \ .$$

We can use the comparison test see that the integral diverges.

We can fill the trumpet with paint but can not paint its surface!