CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 8: Trapezoid Rule

Numerical methods overview

Definition: For a fixed division $a = x_0, \dots, x_n = b$ of the interval $[a, b]$ into slices of size $\Delta x = (b-a)/n$ we have the **left Riemann sum**:

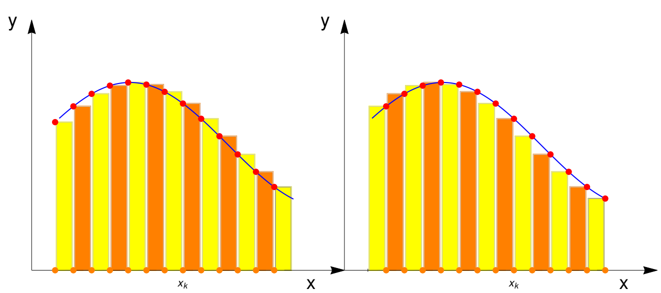
$$L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x \ ,$$

the **right Riemann sum**

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

and the **trapezoid rule**:

$$T_n = (L_n + R_n)/2$$



Definition: The **Midpoint rule** sums up the value at the center $y_k = (x_k + x_{k+1})/2$ of the slice

$$M_n = \sum_{k=1}^{n} f(y_k)\Delta x \ .$$

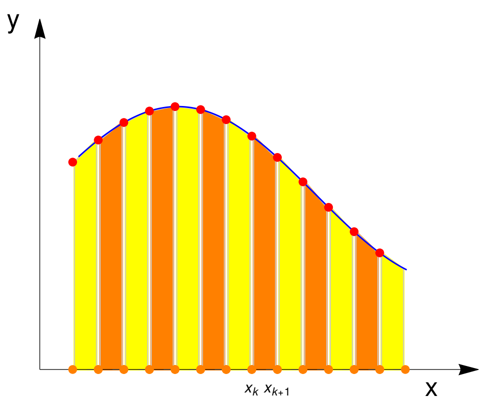
8.1. We have the same bound for the midpoint rule than for the left or Riemann sum:

$$|M_n - I| \leq M_{(1)} \frac{(b-a)^2}{2n} \ .$$

Definition: The **Simpson rule** averages the midpoint and left and right sums in a clever way:

$$S_n = \frac{(b-a)}{6} \sum_{k=1}^{n} [f(x_k) + 4f(y_k) + f(x_{k+1})]\Delta x \ ,$$

where y_k again is the midpoint between x_k and x_{k+1}. We have $\boxed{S_n = (2M_n + T_n)/3}$.



8.2. The Trapezoid rule is exact for linear functions. The Simpson rule is exact for quadratic functions. For a general quadratic function $f(x) = Ax^2+Bx+C$, the formula

$$\int_a^b f(x)\, dx = [f(a) + 4f((a+b)/2) + f(b)](b-a)/6$$

holds exactly. Already Kepler was using the Simpson rule.

Error

8.3. The error bound for left and right Riemann sums are

$$|R_n - I| \leq M_{(1)} \frac{(b-a)^2}{2n}$$

where $M_{(1)}$ is an bound for $|f'(x)|$ on $[a,b]$.

8.4. The error bound for the trapezoid rule is

$$|T_n - I| \leq M_{(2)} \frac{(b-a)^3}{12n^2}$$

where $M_{(2)}$ is an bound for $|f^{(2)}(x)|$ on $[a,b]$. (To remember the number, note that $12 = 2!3!$.)

8.5. The error bound for Simpson is:

$$|S_n - I| \leq M_{(4)} \frac{(b-a)^5}{2880n^4}$$

where $M_{(4)}$ is an upper bound for the fourth derivative $|f^{(4)}(x)|$ on $[a,b]$. (To remember the number, note that $2880 = 4!5!$.)

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