CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 7: Numerical Integration

Numerical methods

7.1. Numerical methods were used long before computers have been available. The goal is to get solutions to integration problems, even if an analytic solution is missing. Early motivations were astronomical tasks, or the task to compute volumes of bodies. The Simpson method mentioned here was already used by Johannes Kepler. His **Fass-regel** allowed to compute the volume of wine barrel as the height h times an average of the cross sections A, B at both ends and the center C. Kepler got $h(A+4C+B)/6$, which is the Simpson method. He noticed in his work **Nova Stereometria doliorum vinariorum** that the formula gives even exact results for pyramids, sphere, elliptical paraboloids or hyperboloids.

Left and Right Riemann sum

7.2. The integral $\int_a^b f(x)\, dx$ can be evaluated numerically by Riemann sums:

Definition: For a fixed division $x_0, \dots, x_n$, the sum $L = \sum_{k=0}^{n-1} f(x_k)\Delta x$ is called the **left Riemann sum** and $R = \sum_{k=1}^{n} f(x_k)\Delta x$ is called the **right Riemann sum**.

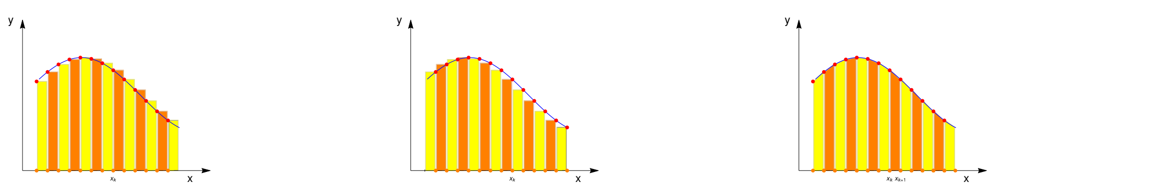

Figure: Left Riemann sum L and right Riemann sum R and the trapezoid rule $(L+R)/2$.

Trapezoid rule

Definition: The average $T = (L+R)/2$ between the left and right hand Riemann sum is called the **Trapezoid rule**. Geometrically, it sums up areas of trapezoids instead of rectangles.

The trapezoid rule does not change things much as it sums up almost the same sum. For the interval $[0,1]$ for example, with $x_k = k/n$ we have

$$R - L = \frac{1}{n}[f(1) - f(0)] \; .$$

Simpson rule

Definition: The **Simpson rule** computes the sum

$$S_n = \frac{1}{6}\sum_{k=1}^{n}[f(x_k) + 4f(y_k) + f(x_{k+1})]\Delta x \; ,$$

where $y_k = (x_k + x_{k+1})/2$ is the midpoint between x_k and x_{k+1}.

7.3. The Simpson rule gives the actual integral for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u}\int_u^v f(x)\;dx = [f(u) + 4f((u+v)/2) + f(v)]/6$$

holds exactly.

7.4. With a bit more calculus one can show that for smooth functions the Simpson rule is n^{-4} close to the actual integral. For 100 division points, this can give accuracy to 10^{-8} already.

Monte Carlo

7.5. If we choose n random points x_k in $[a,b]$ and look at the sum divided by n we get the **Monte Carlo method**.

Definition: The **Monte Carlo** integral is the limit S_n to infinity

$$S_n = \sum_{k=1}^{n} f(x_k)\Delta x \; ,$$

where x_k are n random points in $[a,b]$ and $\Delta x = (b-a)/n$.

The Monte Carlo integral is equivalent to the **Lebesgue integral**.
It works also for functions which are crazy like the function on $[0,1]$ which gives 1 if x is rational and 0 else. The Riemann integral would give 1. The Monte-Carlo integral gives 0 because almost all points are 0.

Error

7.6. We will see in the lecture and worksheet that the error for left or right Riemann sums is $\boxed{M(b-a)^2/(2n)}$, where M is a bound for the maximal absolute value of the derivative of f in (a,b).