CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 4: Integration techniques

Know some integrals

4.1. There are a few functions for which you should just **know** the anti-derivative. It is like in chemistry. A few molecules like water or methane should be known without having to look it up. Functions to know to integrate are polynomials like x^5 or $\sqrt{x}$, rational functions like $1/x$ or $1/(1+x^2$ or trig functions like $\sin, \cos, \tan$ and the exponential exp and the logarithm log. Maybe it is also good to recall the derivatives of inverse trig functions to tackle functions like $1/\sqrt{1-x^2}$.

Substitution

4.2. To substitute, identify part of the formula as u, then differentiate it to get du in terms of dx, then replace dx with du. Example:

$$\int \frac{x}{1+x^4} \, dx \; .$$

Solution: Substitute $u = x^2, du = 2xdx$ gives $(1/2) \int du/(1+u^2) \, du = (1/2) \arctan(u) = (1/2) \arctan(x^2) + C$.

Remarks. Sometimes you need to try multiple times. In the example, we might first try $u = 1 + x^4$ but that does not give the desired cancellation. If you should forget substitution, remember the **chain rule**. If $f(x) = g(u(x))$ then $f'(x) = g'(u(x))u'(x)$. Look out for a function $u(x)$ so that you find both $u(x)$ and $u'(x)$.

Integration by parts

4.3. Write the integrand as a product of two functions, differentiate one u and integrate the other dv. Then use $\int udv = uv - \int vdu$ from the product formula.

Example:

$$\int x \cos(x/3) \, dx$$

Solution: differentiate $u = x$ and integrate $dv = \cos(x/3)dx$. We have $3x \sin(x/3) - \int 1 \cdot 3 \sin(x/3) = 3x \sin(x/3) + \cos(x/3)9 + C$.

Remarks. If you should forget the rule, remember the product rule $d(uv) = udv + vdu$ and integrate it, the solve for $\int u \, dv$.

Partial fractions

4.4. Algebra allows to write a fraction as a sum of simpler fractions. Example:

$$\int \frac{1}{(x-3)(x-2)}\, dx$$

Solution: write $\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$.
To get A, multiply with $x-3$, cancel terms and put $x=3$ which gives $A=1$. To get B, multiply with $x-2$, cancel terms and put $x=2$ which gives $B=-1$.

Trig substitution

4.5. Replace a term with $\sin(u)$ so that the formula simplifies.

Example: A prototype example is

$$\int \frac{1}{\sqrt{64-x^2}}\, dx \ .$$

Solution: $x = 8\sin(u)$ gives $\frac{1}{\sqrt{64-x^2}} = 1/(8\cos(u))$. As $dx = 8\cos(u)$. The integral is $\int 1/8 du = u/8 + C = \arcsin(x/8) + C$.

Trig identities

4.6. The double angle formulas $\cos^2(x) = (1+\cos(2x))/2$ and $\sin^2(x) = (1-\cos(2x))/2$ are handy. Also consider using $\cos^2(x) = 1-\sin^2(x)$ or $\sin^2(x) = 1-\cos^2(x)$ or use the identity $2\sin(x)\cos(x) = \sin(2x)$.

Example:

$$\int \sin^4(x)\, dx = \int (1-\cos^2(x))\sin^2(x) = \int \sin^2(x) - \sin^2(2x)/4\, dx$$

we can now use the double angle formulas to write this as $\int (1-\cos(2x))/2 - (1-\cos(4x))/8$ which now can be integrate $x/2 - \sin(2x)/4 - x/8 + \sin(4x)/32 + C$.

Symmetries

4.7. Sometimes, the result of an integral can be seen geometrically.

Example:

$$\int_{-2}^{2} \sin^7(5x^3)\, dx$$

can not be compute so easily because we can not write down an explicit anti-derivative. However we see that the function is odd $f(-x) = -f(x)$. If we integrate an odd function over a symmetric interval, we have a cancellation of areas. The answer is 0.

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