CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 1: Density and approximation

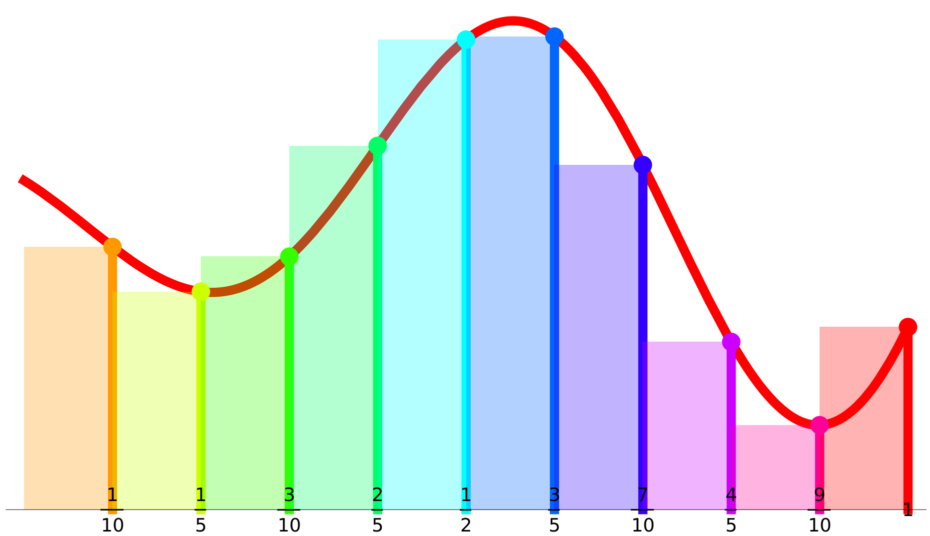
Density and approximation

1.1. We are given a function $f(x)$ which is a **density function**. To integrate up the density along an interval $[a, b]$, we can **slice it** and get a **Riemann sum**:

$$\sum_{k=1}^{n} f(x_k)\Delta x \ ,$$

where $\Delta x = (b-a)/n$ is the slice size and $x_k = a + k\Delta x$. In the limit when the spacing goes to zero, we will reach the integral

$$\int_a^b f(x)\ dx \ .$$



1.2. As a general rule, as finer the spacing is made, as better we **approximate** the value of the integral Using slicing, we can get **upper bounds** and a **lower bounds** of the sum.

1.3. In the first lecture, we look at finite sums

$$\sum_{k=1}^{n} \rho(x_k) f(x_k) \Delta x \ ,$$

where $\rho(x)$ is a **density**. It leads to the **integral**

$$\int_a^b \rho(x) f(x) \ dx \ .$$

1.4. What happens if we include the **density function** is that we **weight** different slices in a different way. We see in worksheets or homework many different examples.

1.5. The function ρ could be the **Parmesan cheese density** on a pizza, the **ink density** of a printer, the **population density** of a town, the **mass density** of a wire, a **charge density** or a **probability density**.

1.6. Why do we not just combine the two functions $\rho(x)$ and $f(x)$ to one single function? One reason is that in many applications, the information is given as a density problem where the function $f(x)$ depends on the geometry

1.7. An important special case is if $\rho(x) = 1$. In that case $\int_a^b f(x) \ dx$ can be interpreted as an **area**. If $f(x) \geq 0$, the integral above represents the area between the graph of f and the x-axis.

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