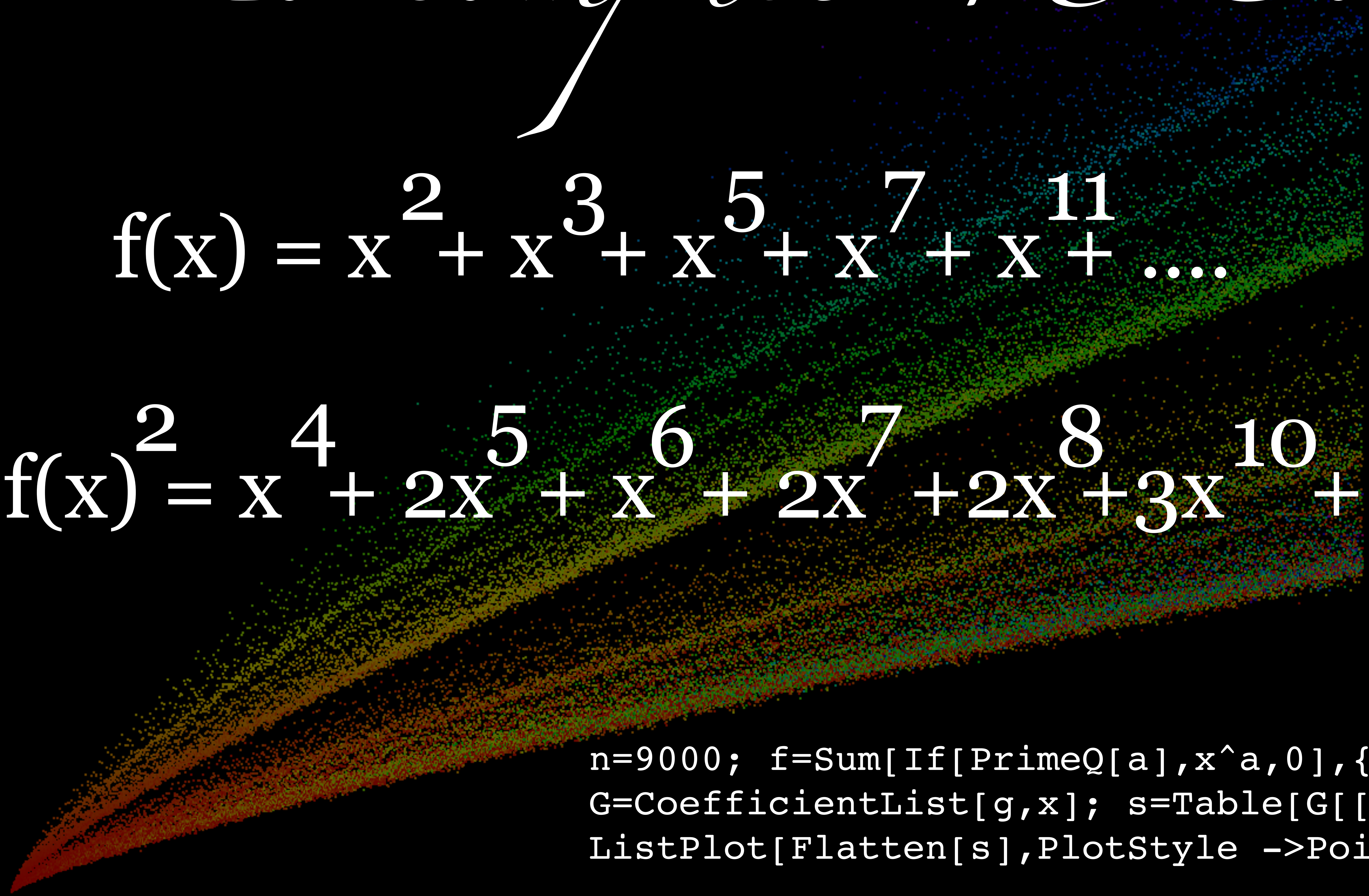
# Goldbach Comet

Oliver Knill

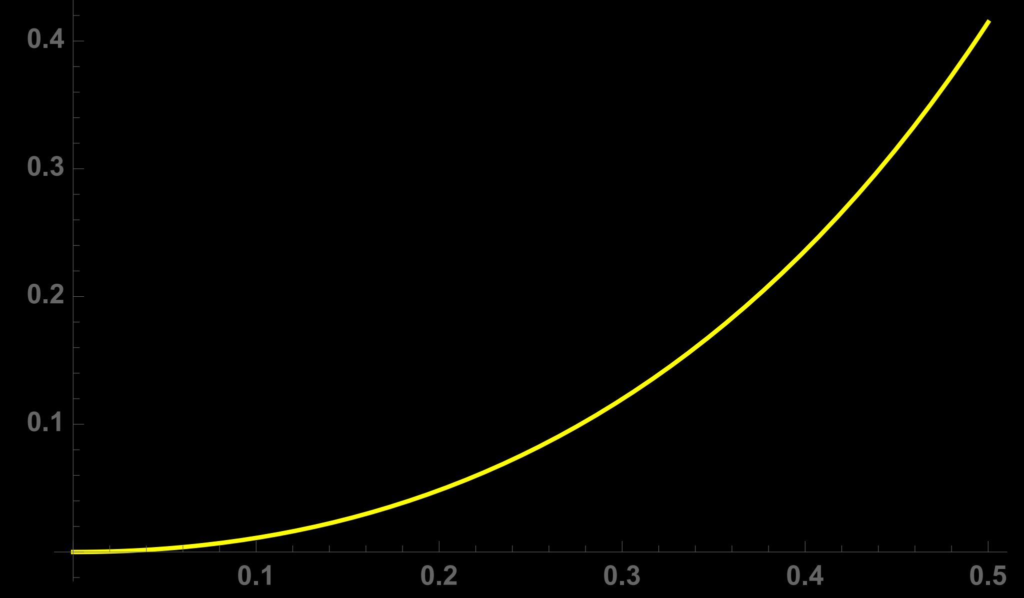
Math Table, 10/23/2018

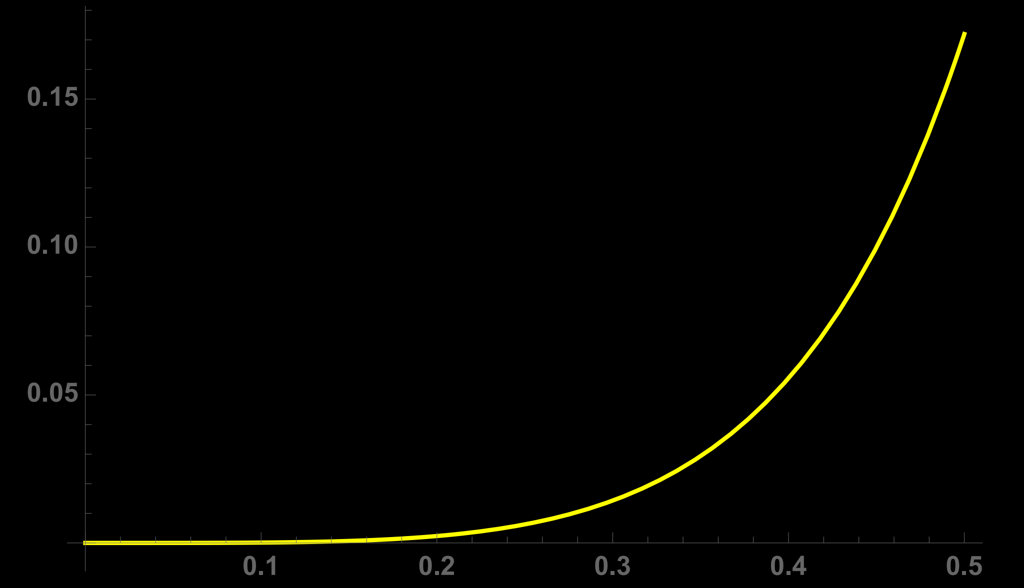
# *To compute the Comet*



$$f(x) = x^2 + x^3 + x^5 + x^7 + x^{11} + \dots$$

$$f(x)^2 = x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 3x^{10} + \dots$$




```
n=9000; f=Sum[If[PrimeQ[a],x^a,0],{a,n}]; g=Expand[f*f];
G=CoefficientList[g,x]; s=Table[G[[2 k-1]],{k,2,n/2}];
ListPlot[Flatten[s],PlotStyle ->PointSize[0.001]]
```

# Edmund Landau 1912


Goldbach

Prime Twin

Legendre

Landau

# From Proceedings


3. Professor E. Landau.

Familiarity with the theory of numbers is by no means general among mathematicians; in particular, the difficulties of the analytic theory of numbers have not been attractive, so that but few are familiar with the elegant results of the theory.

After stating the problem of the lecture, a rapid resumé of preceding results and methods was given, and a concise statement of several new results, particularly those of Littlewood, Bohr, and Landau, was added.

Four definite questions were put, the solutions of which were considered as impossible in the present state of the science.

1. Does the function $u^2 + 1$, $u$ an integer, represent an infinite number of primes?

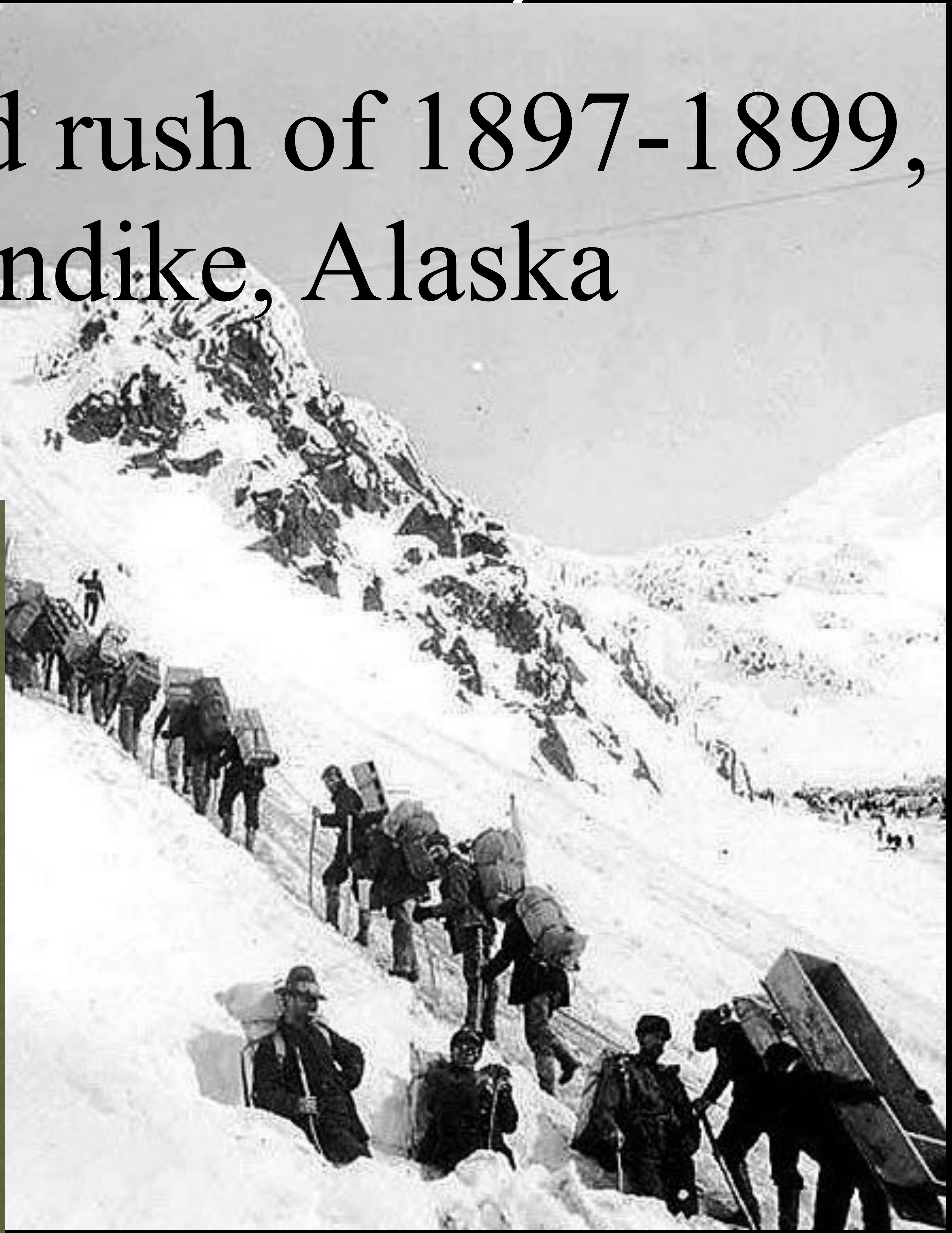
2. Does the equation $m = p + p'$ have a solution in prime numbers for every even value of $m$?

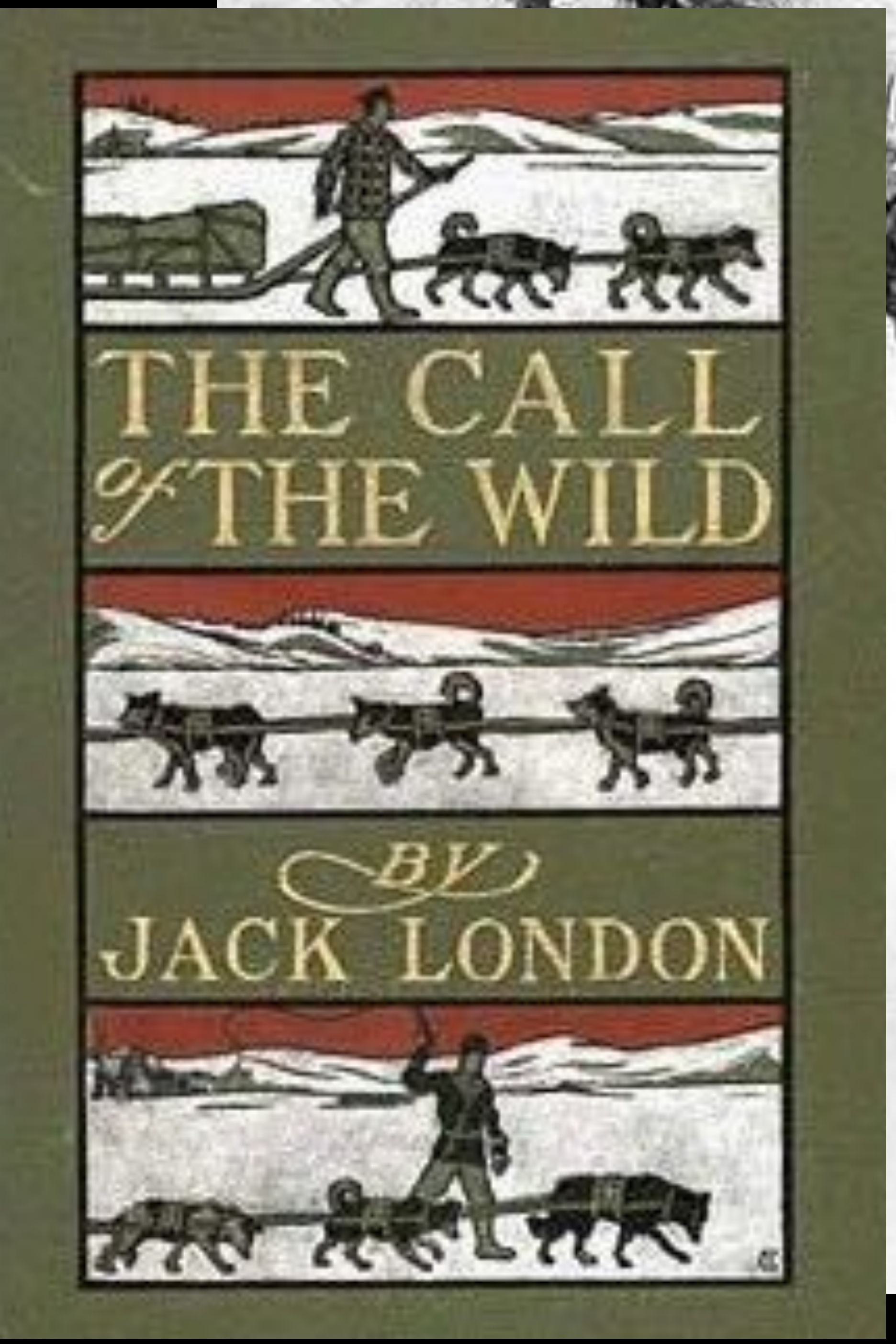
3. Has the equation $2 = p - p'$ an infinite number of prime solutions?

4. Does at least one prime number lie between $n^2$ and $(n + 1)^2$ for every integral value of $n$?

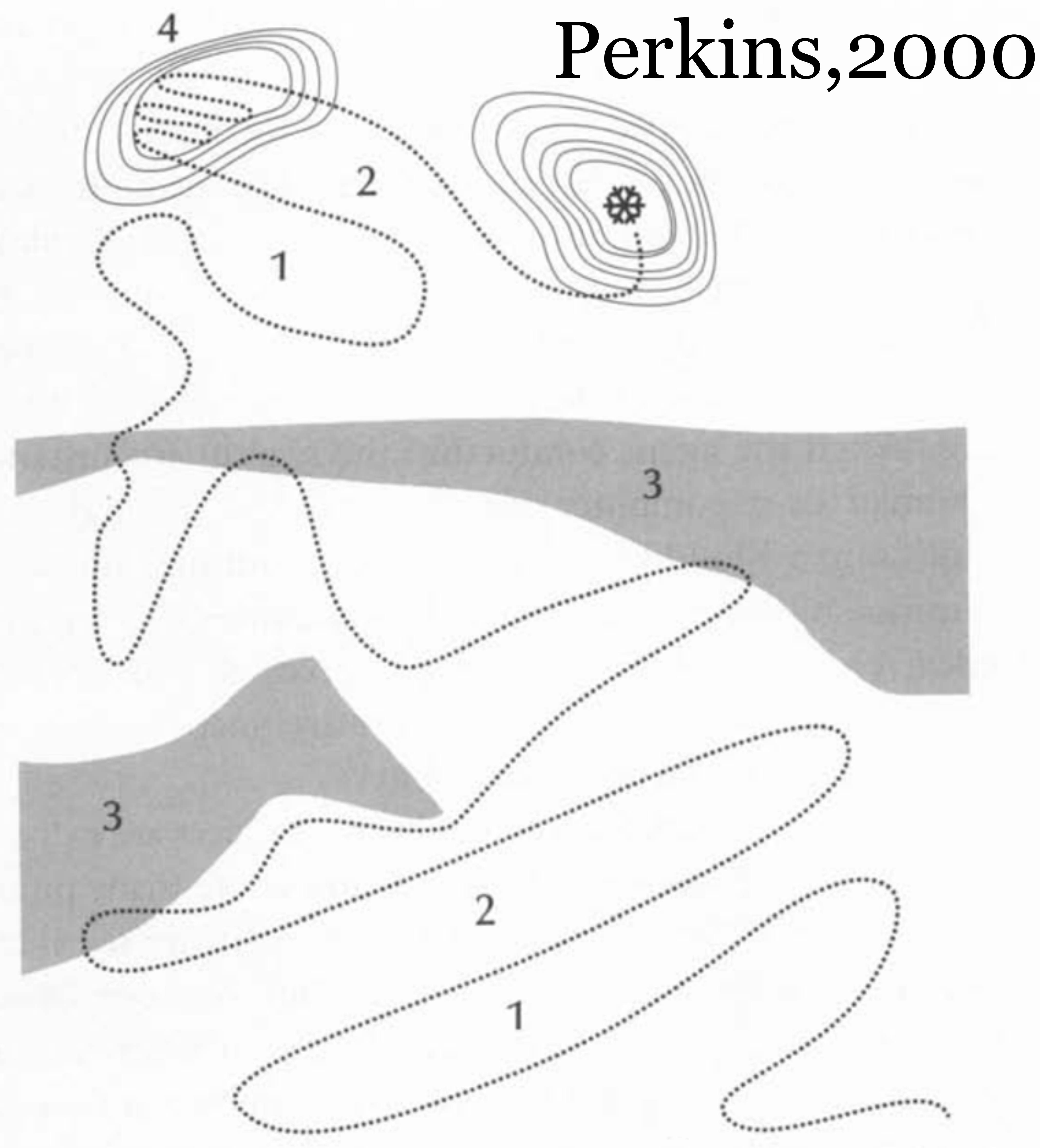
# Klondike Space

gold rush of 1897-1899,
Klondike, Alaska






Perkins,2000




Search in a Klondike Space: 1. A large space with few solutions (a wilderness trap). 2. Regions with no clues pointing direction (plateau traps). 3. A barrier isolates the solution (creating a canyon trap). 4. An area of high promise but no solution (an oasis trap).

*Apropos old times …*

# Once upon a time ...


# some school friends

*told me about*

# Prime Numbers

An Integer N Larger Than 1 divisible only by 1 and N

2,3,5,7,11,13,17,....

# *Today, kids learn it from holywood*

CONTACT, 1997

*As for Goldbach…*


I had to
wait for high school
UNIVERSITY
FLORIDA

# 2 Kbytes RAM, 16 KB ROM

*But also exciting*


Olivetti P6040, 1977

16 character diode display!

*And intimidating*

Translated from the Russian
by HELEN POPOVA, Ph.D.
University of Aberdeen

# An Introduction to THE THEORY OF NUMBERS

by I. M. VINOGRADOV
Member of the Academy of Sciences, U.S.S.R.
For. Mem. R.S.

PERGAMON PRESS
LONDON & NEW YORK
1955



**13.** (a) For $R(s) > 1$, prove that

$$\zeta(s) = \prod \frac{1}{1 - \frac{1}{p^s}}$$

where $p$ runs through all prime numbers.

(b) Prove the infinitude of prime numbers by using the fact that the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is divergent.

(c) Prove the infinitude of prime numbers by using the fact that $\zeta(2) = \frac{\pi^2}{6}$ is irrational.

**14.** Denoting by $\Lambda(a)$ the function equal to $\log\, p$ when $a = p^l$, where $p$ is prime and $l$ is a positive integer, and zero for all other positive integers $a$, prove the identity

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \text{where} \quad R(s) > 1.$$

**15.** Let $R(s) > 1$. Prove that

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

where $p$ runs through all prime numbers.

**16.** (a) Let $n > 1$. By using **D**, § **3**, show that

$$1 = \sum_{0<d\leqslant n} \mu(d) \left[\frac{n}{d}\right]$$

(b) Let

$$M(z, z_0) = \sum_{z_0<a\leqslant z} \mu(a); \quad M(x) = M(x, 0).$$

Prove that

$$(\alpha) \quad M(n) + M\left(\frac{n}{2}\right) + M\left(\frac{n}{3}\right) + \ldots = 1, \quad n \geqslant 1.$$

$$(\beta) \quad M\left(n, \frac{n}{2}\right) + M\left(\frac{n}{3}, \frac{n}{4}\right) + M\left(\frac{n}{5}, \frac{n}{6}\right) + \ldots = 1, n \geqslant 2.$$

(c) Let $n \geqslant 1$, let $l > 1$ be an integer, and let $T_{l,n}$ denote the number of integers $x$, satisfying $0 < x \leqslant n$, which are not divisible by an $l$th power distinct from 1. By using **D**, § **3**, prove that

$$T_{l,n} = \sum_{d=1}^{\infty} \mu(d) \left[\frac{n}{d^l}\right].$$

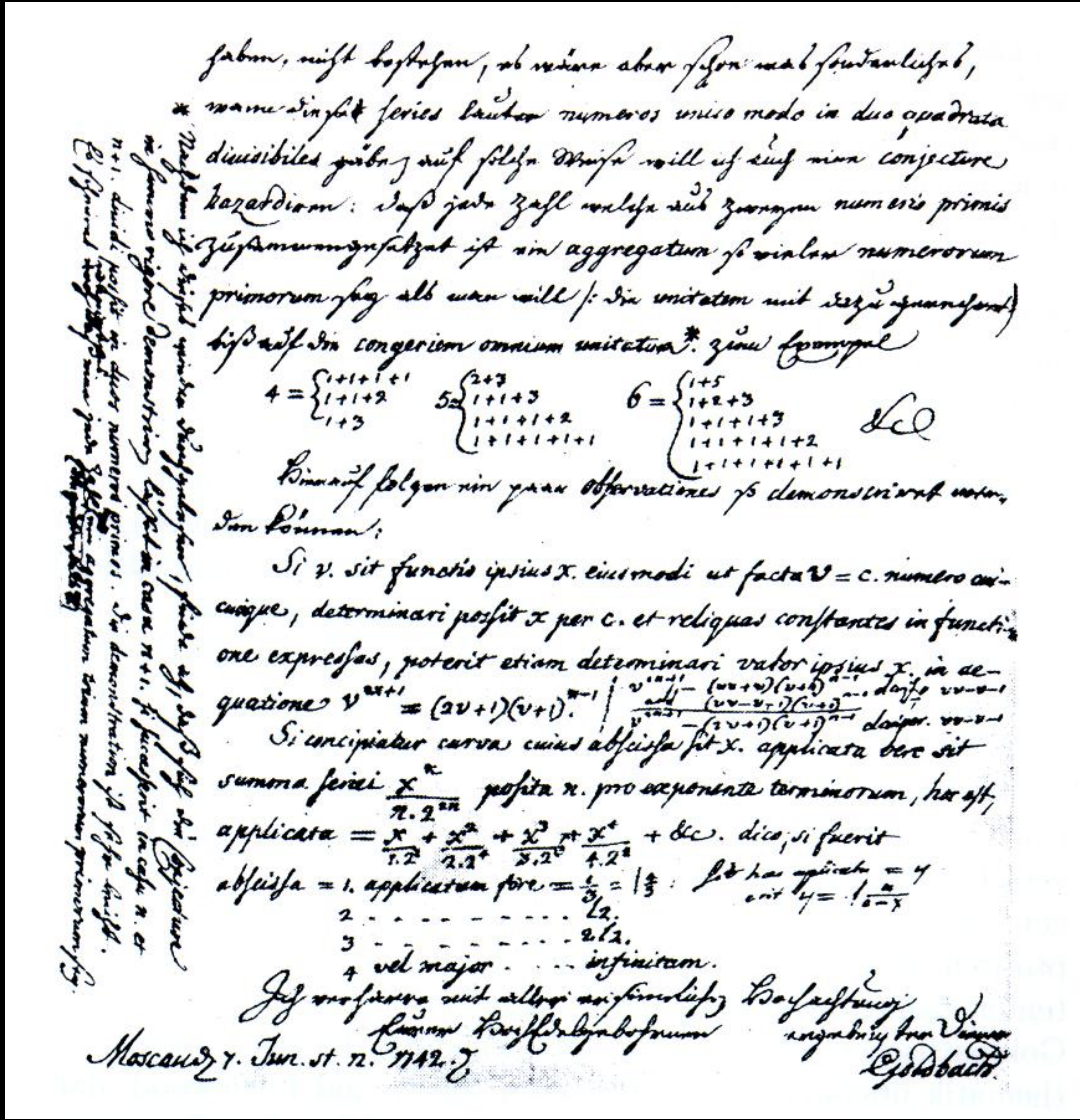
**17.** (a) Let $a$ be an integer $a > 0$, and let the function $f(x)$ be uniquely defined for integers $x_1, x_2, \ldots, x_n$. Prove that

$$S' = \sum_{d \mid a} \mu(d) S_d$$

where $S'$ denotes the sum of values of $f(x)$ over the values of $x$, relatively prime to $a$, and $S_d$ denotes the sum of values of $f(x)$ over the values of $x$, which are divisible by $d$.

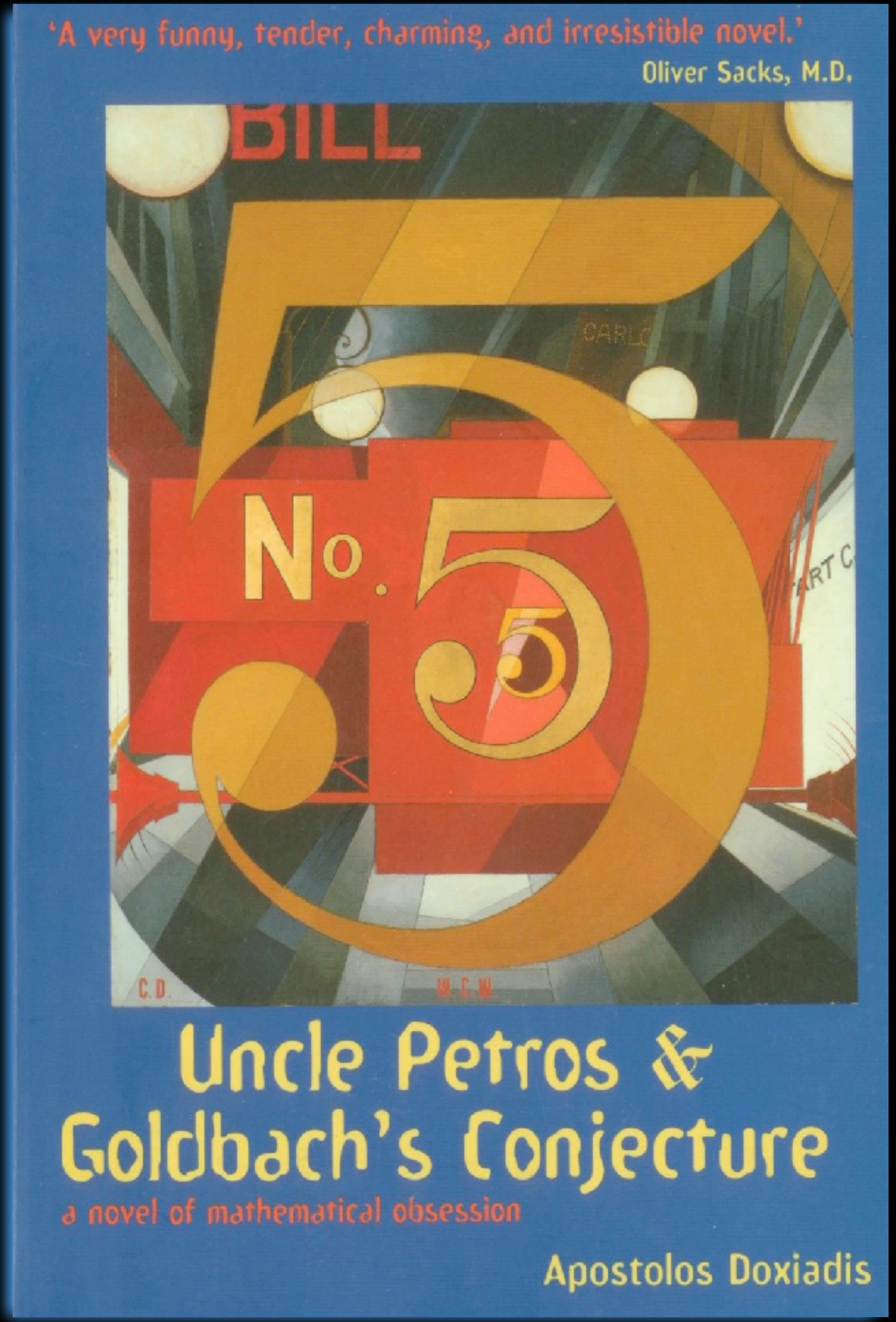
# Christian Goldbach 1742

Every even number larger than 2 is a sum of two primes



# Doxiadis Novel

Uncle Pedros
and the Goldbach
Conjecture (1992)




Hollywood
Calculus of Love (2011)

# Hollywood

## Fermat's Room
## 2007

In 1742, the mathematician
Christian Goldbach

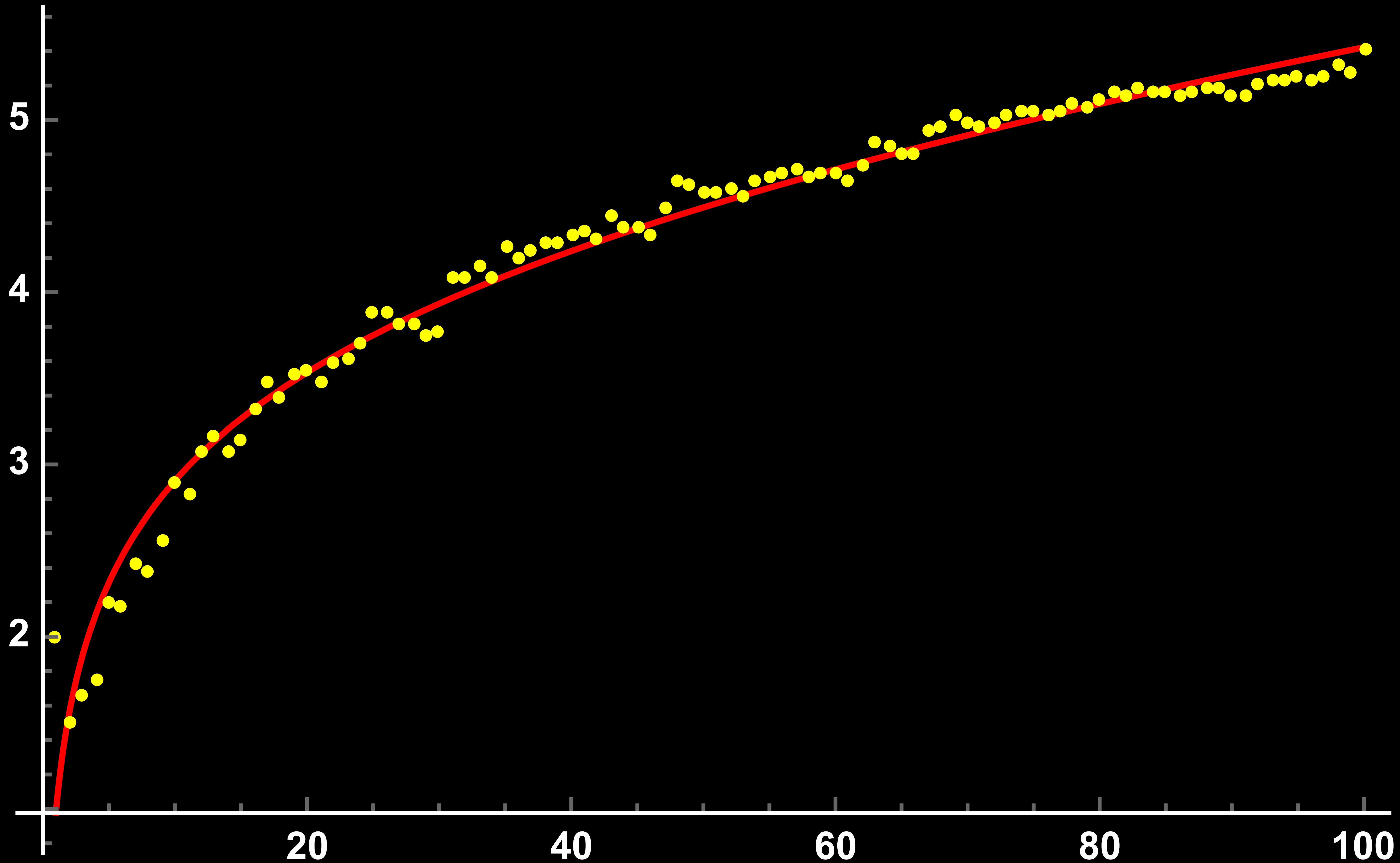
# Fermat's Room
# 2007



To understand
the comet
we need a bit background

# How are Primes Distributed?

# *Experiment (Gauss)*

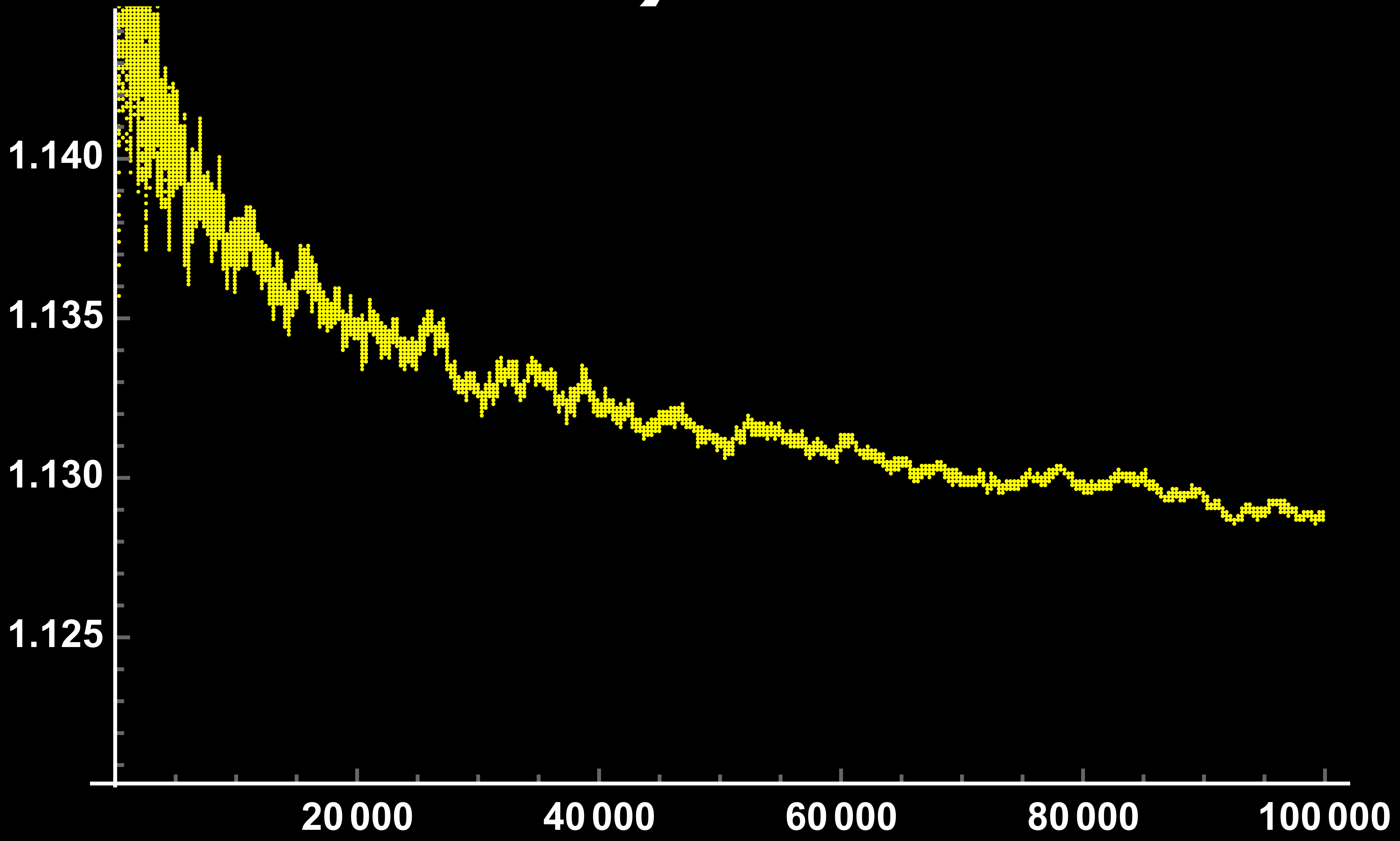



$$\frac{\text{Prime}(k)}{k}$$

grows logarithmically

1801

# *Experiment*




$$\frac{\text{Prime}(k)}{k \log(k)}$$

```
k = 1000000000000;
N[Prime[k]/(k Log[k])]
```

1.0856...

# Prime Number Theorem

The number of primes less or equal to x grows like x/log(x).

1865-1963

Jacques Hadamard and Charles de La Vallee Poussin (1896)

1866-1962

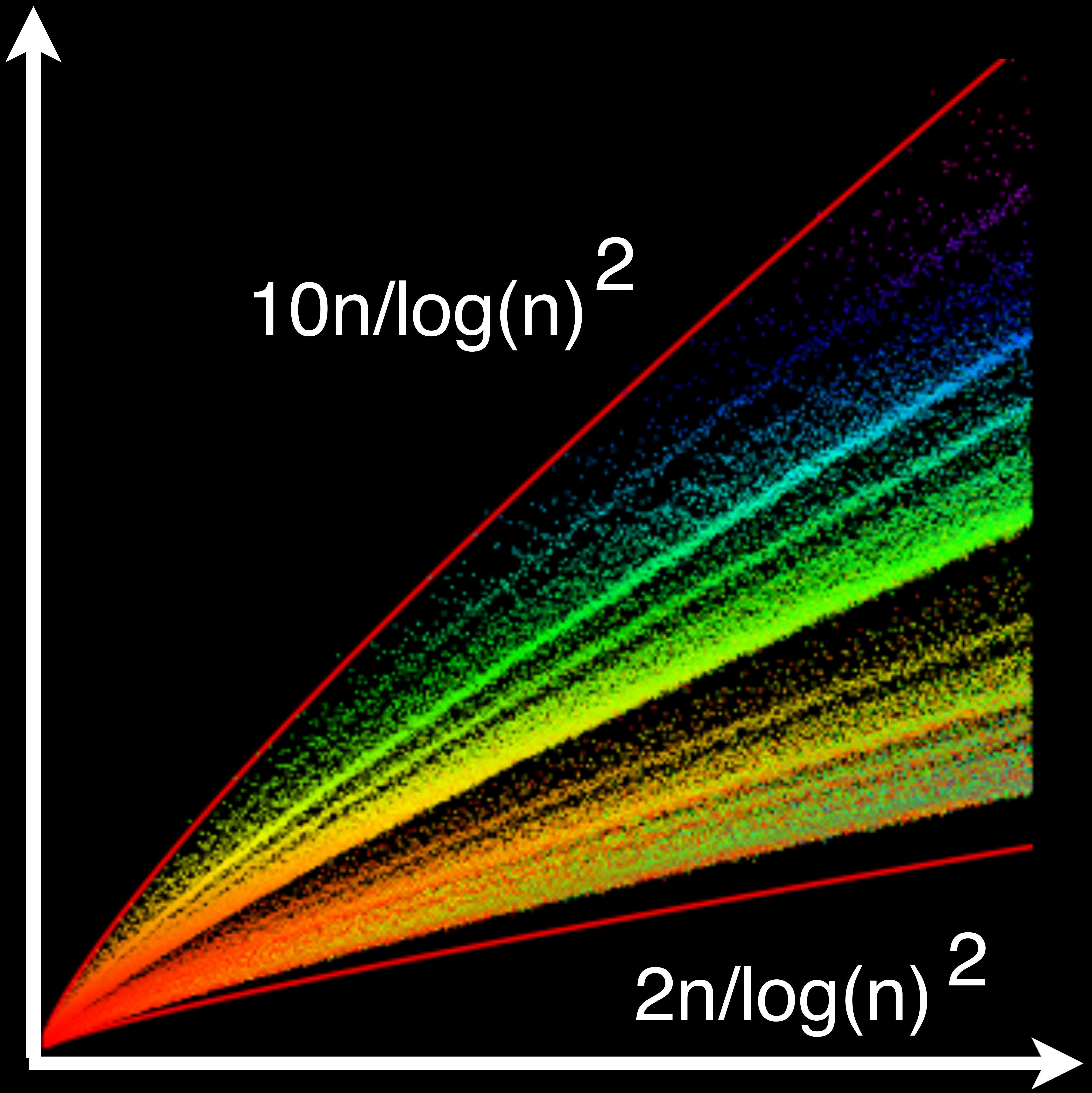
# The Comet

```
M=10000;
h[z_] = Sum[z^Prime[k], {k,M}];
ListPlot[CoefficientList[
  Series[h[z]^2, {z,0,M}], z]]
```

$$\sum_{p+q=2n} \frac{1}{\log(p)} \frac{1}{\log(q)} \approx \frac{n}{\log(n)^2}$$

chance of p to be prime

chance of q to be prime

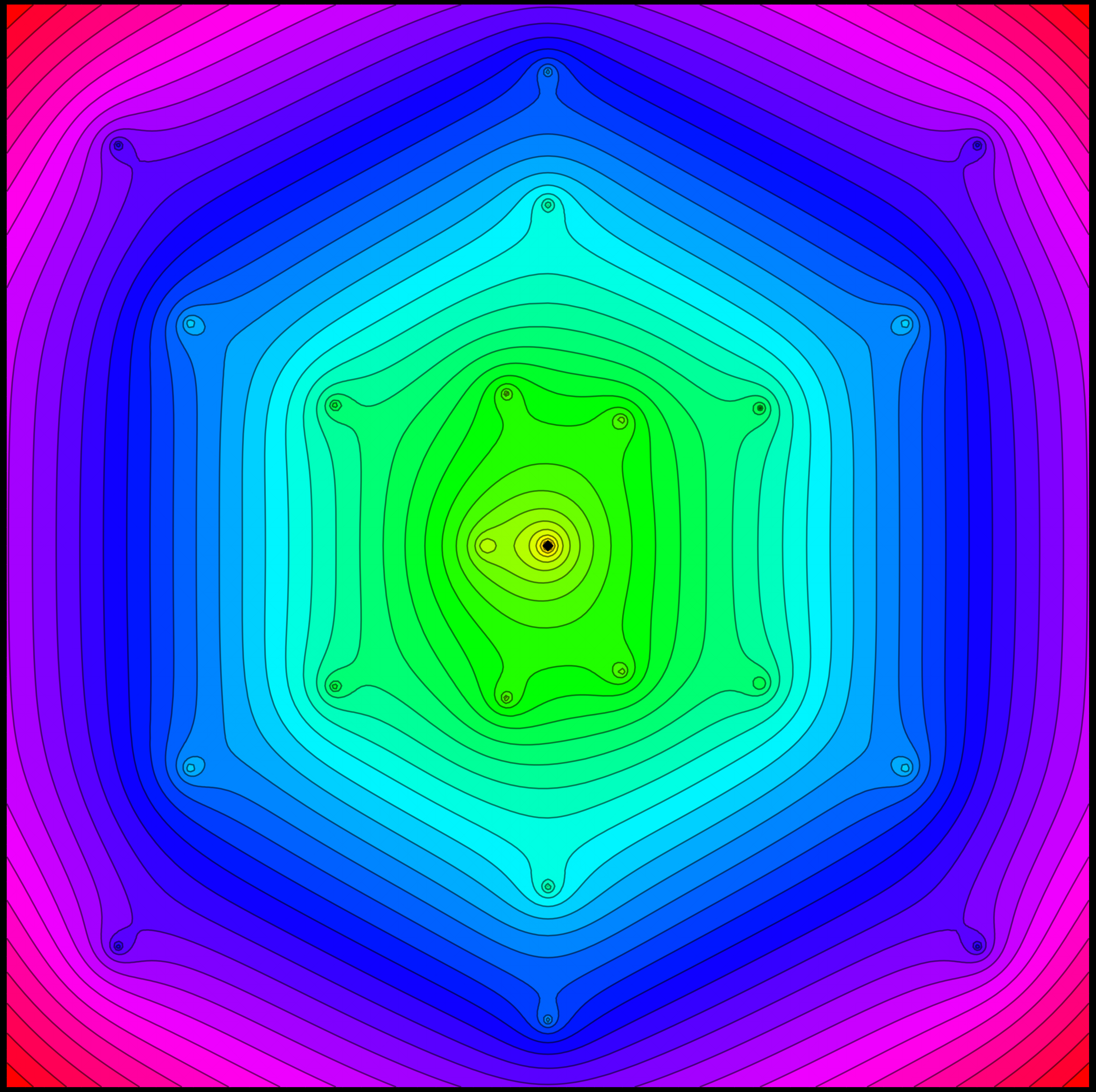


# An other comet

$$f(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \dots$$

# *In the complex*

```
f[z_] := Sum[z^Prime[k] /Prime[k]!, {k,200}];
g[x_, y_] := Log[Abs[N[f[x + I y]^2]]];
(* g[x_, y_] := Abs[f[x + I*y]^2];  *)
S = ContourPlot[g[x, y], {x, -20, 20}, {y, -20, 20}
```

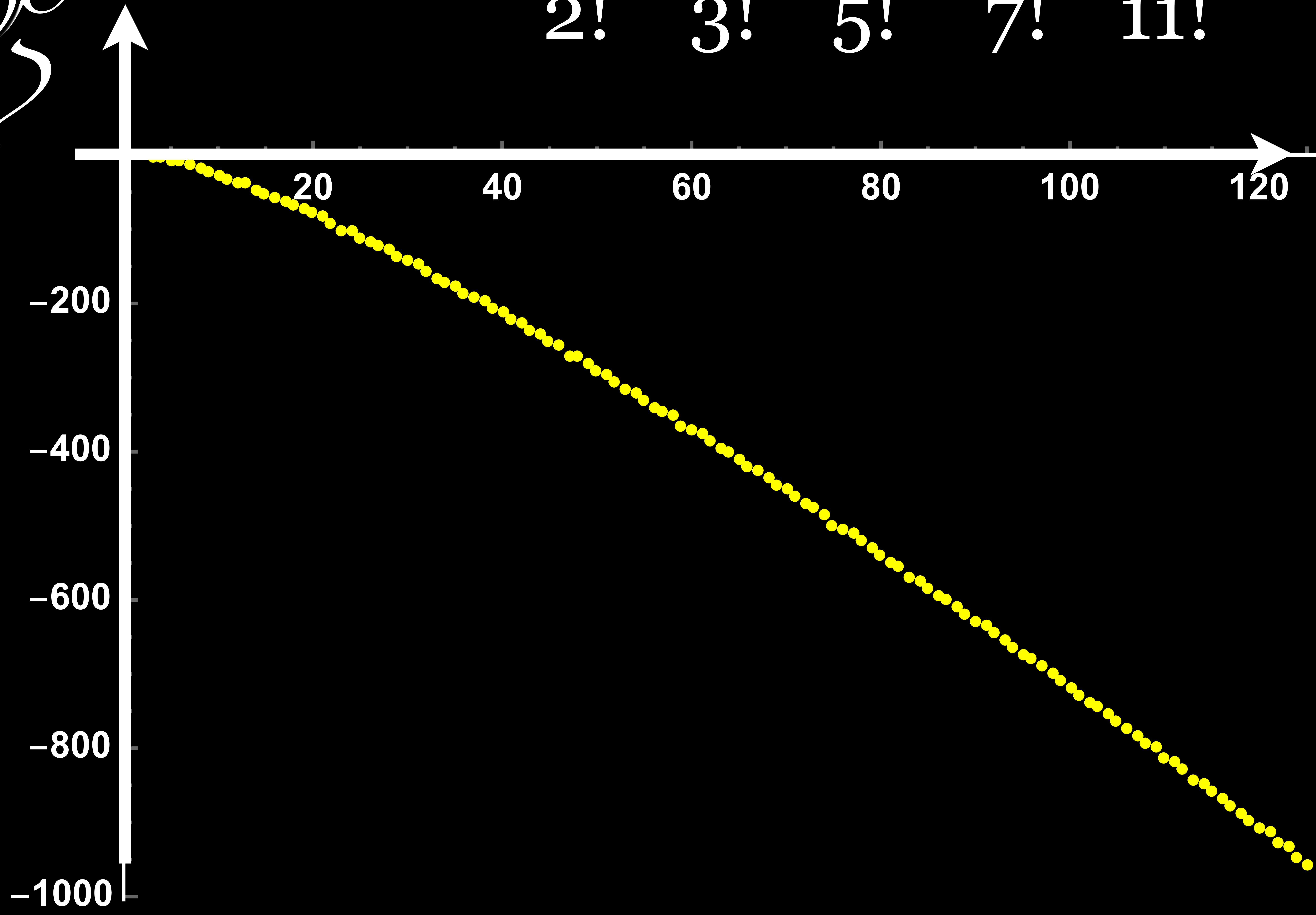


# Learn Complex Analysis!!!

# Math 113

# *Why so smooth?*

$$f(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \dots$$



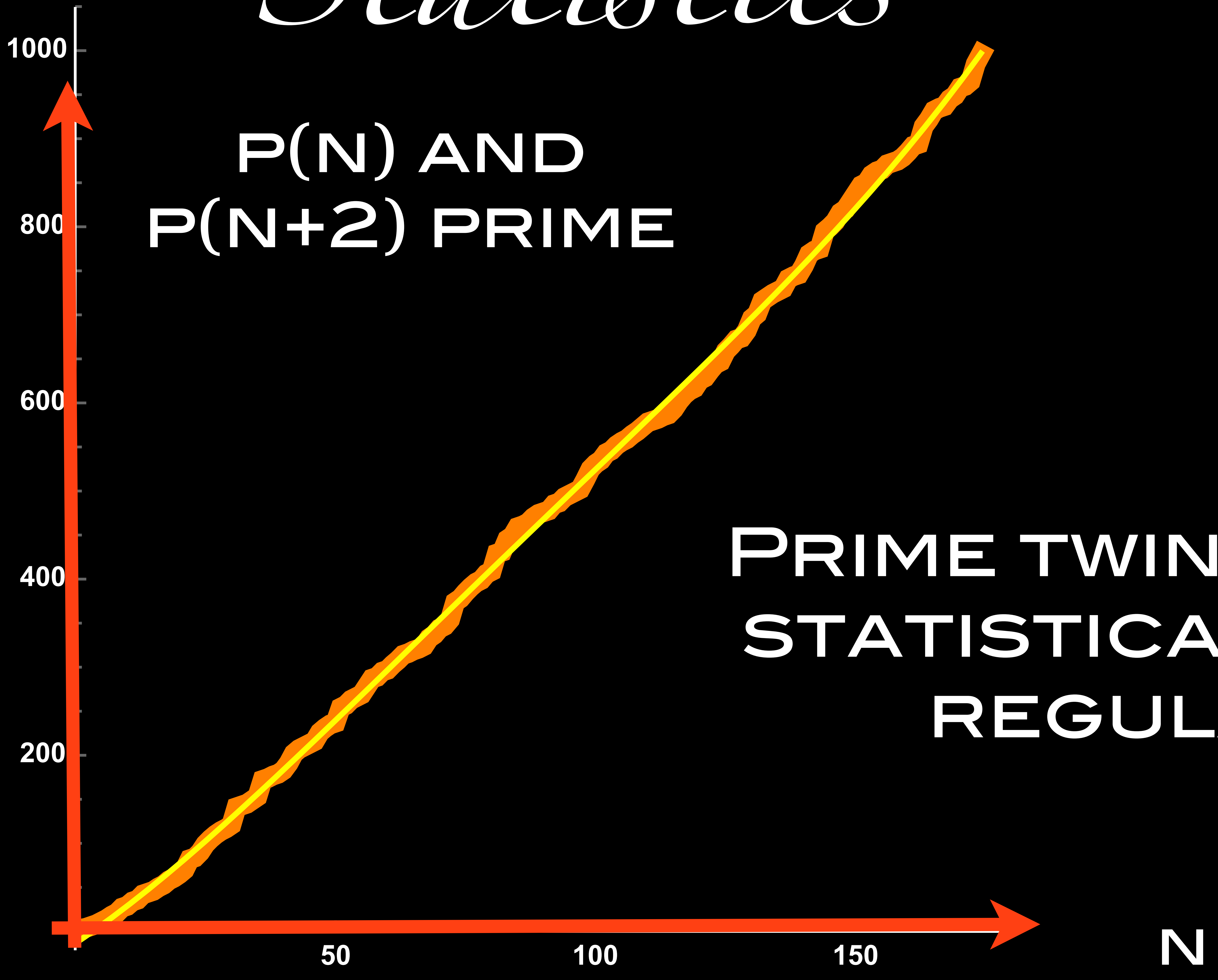
# *Prime Twins*

5

7

Are there infinitely many?

# Statistics




PRIME TWINS APPEAR STATISTICALLY VERY REGULARLY

# Twin Prime Constant

C is The product of p(p-2)/(p-1^2)
where p runs over all
primes is 0.6601618....

P=Prime;Product[P[k] (P[k] - 2)/(P[k] - 1)^2, {k, 2, 1000}]

Hardy-Littlewood: density
of twin primes is believed

$$2\,C\int_2^x \log^2(t)\,dt$$


Hardy and Littlewood
The man who knew infinity, 2015

# *Prime Triplets?*

3 5 7

Are there infinitely many?

# Terry Tao

# Polignac conjecture

"There are infinitely many prime gaps 2n for any n"

Alphonse de Polignac 1849

# *Prime Gaps*

$$\liminf_{n\to\infty}(p_{n+1}-p_n)<7\times10^7$$

*Yitang Zhang, 2013*

Zhang, Yitang. "Bounded gaps between primes"

Annals of Mathematics Pages 1121-1174 from Volume 179 (2014)

# Back to College

ETH Zuerich lecture of Eugene Trubowitz, Photo Oliver Knill

# Kannenberg numbers (A025018)

K(p) = min {2n |2n-q is not prime for all primes q smaller or equal than p}

K(7)=98 as 98-3,98-5,98-7 are not prime

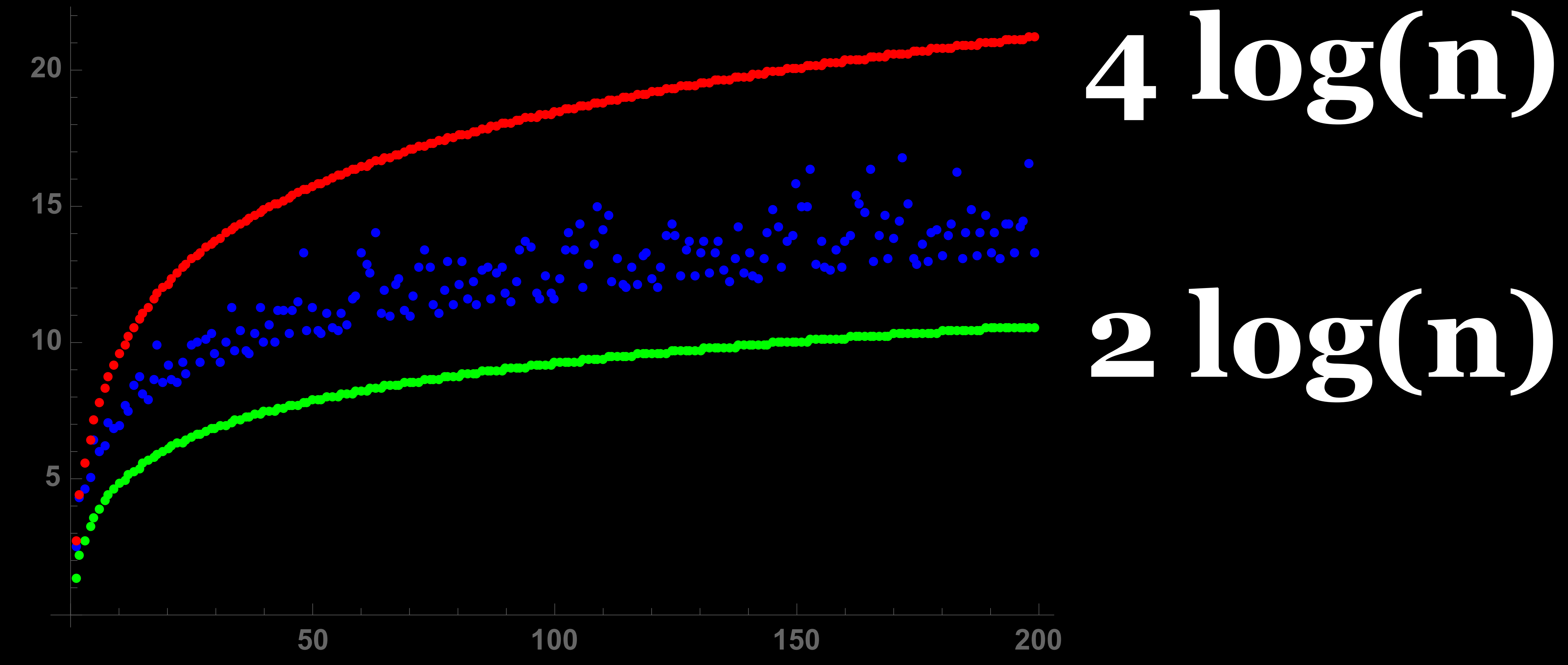
Vax 11/785


8 Meg RAM, 1.2 Gig Disk

# 2007 300 years Euler

$$f(x) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) x^p$$

$$f(1) = \log(\zeta(s))$$

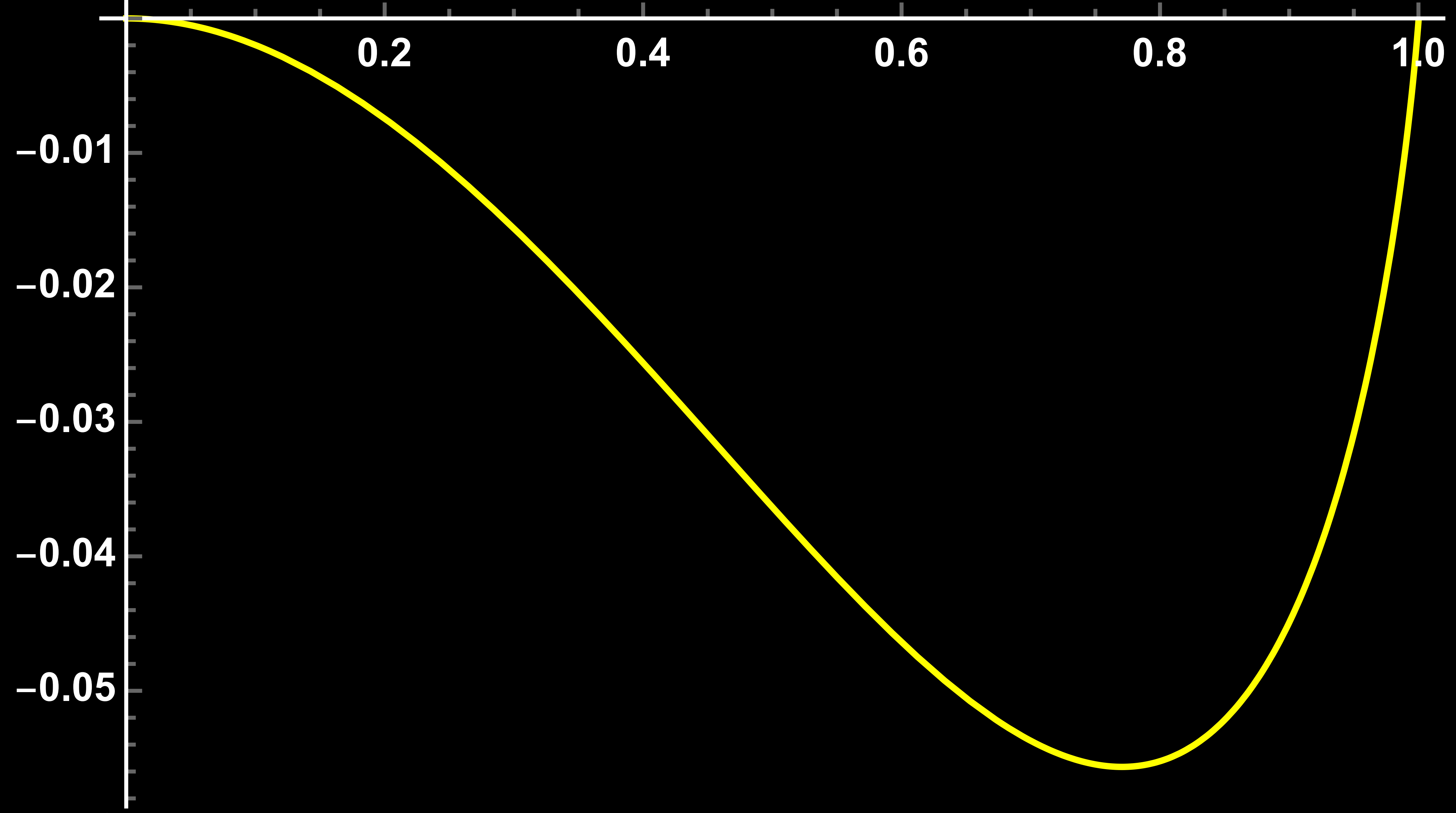


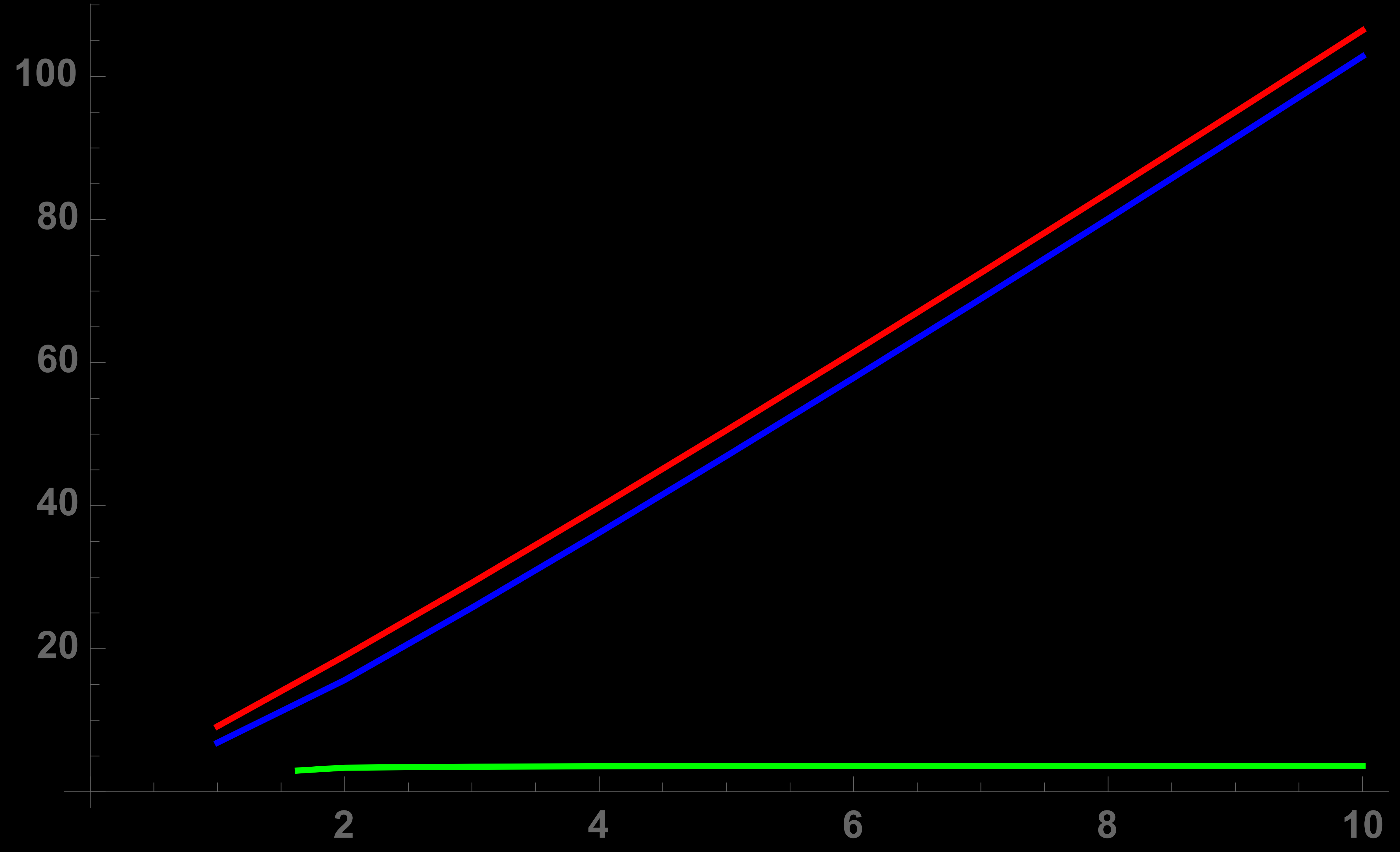
$$g(x)=f(x)-\log(\zeta(2))x^2$$

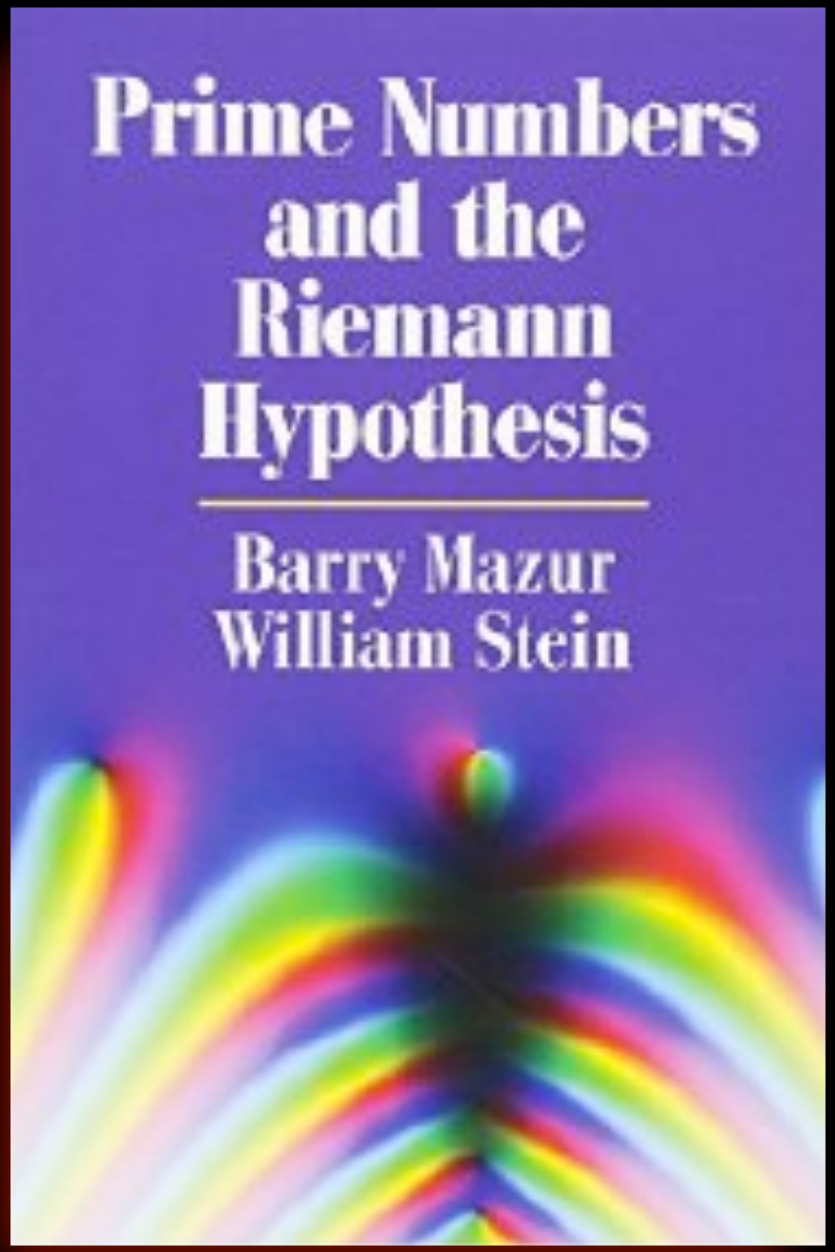
a(n) Fourier Coefficients

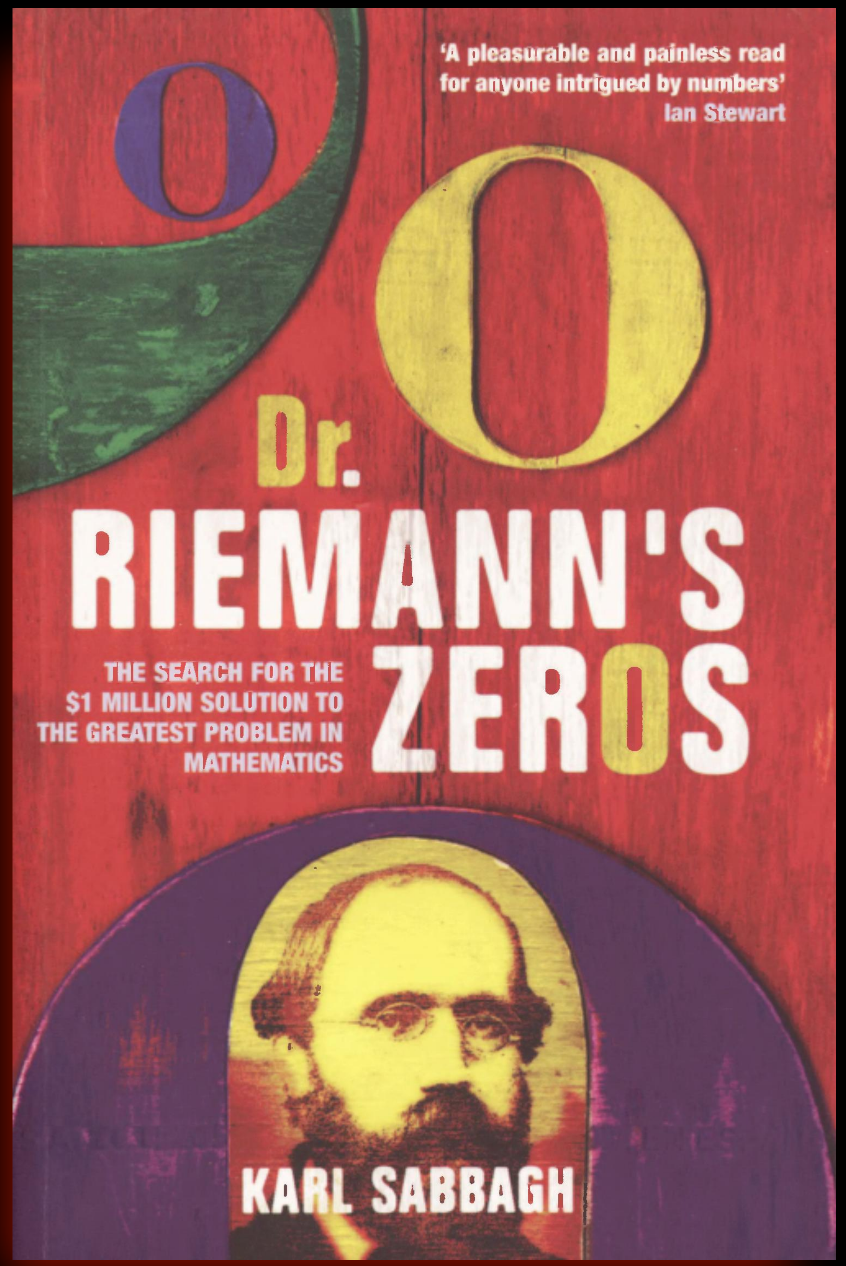
We see $1/\sqrt{a(2n)}$

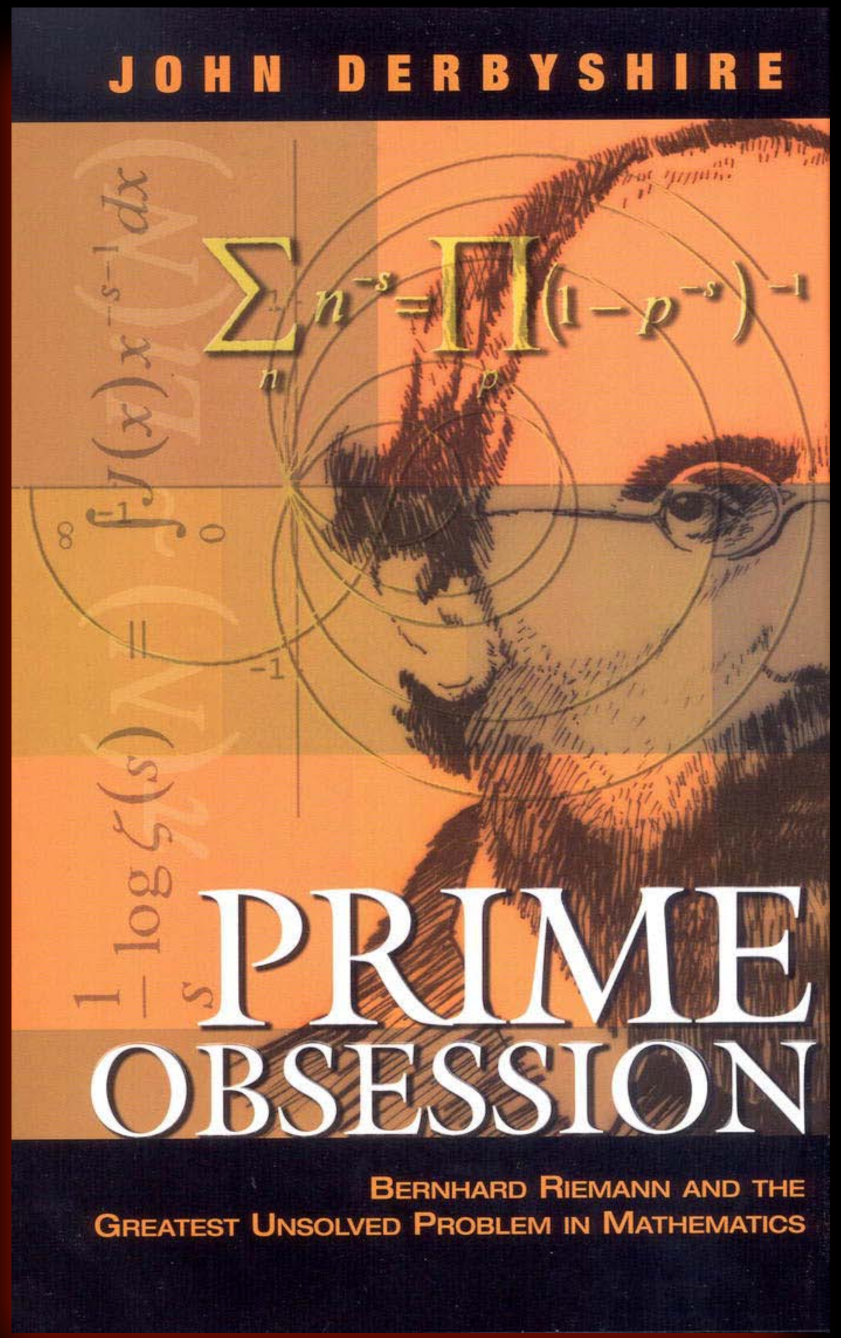
We see $1/\sqrt{a(2n+1)}$

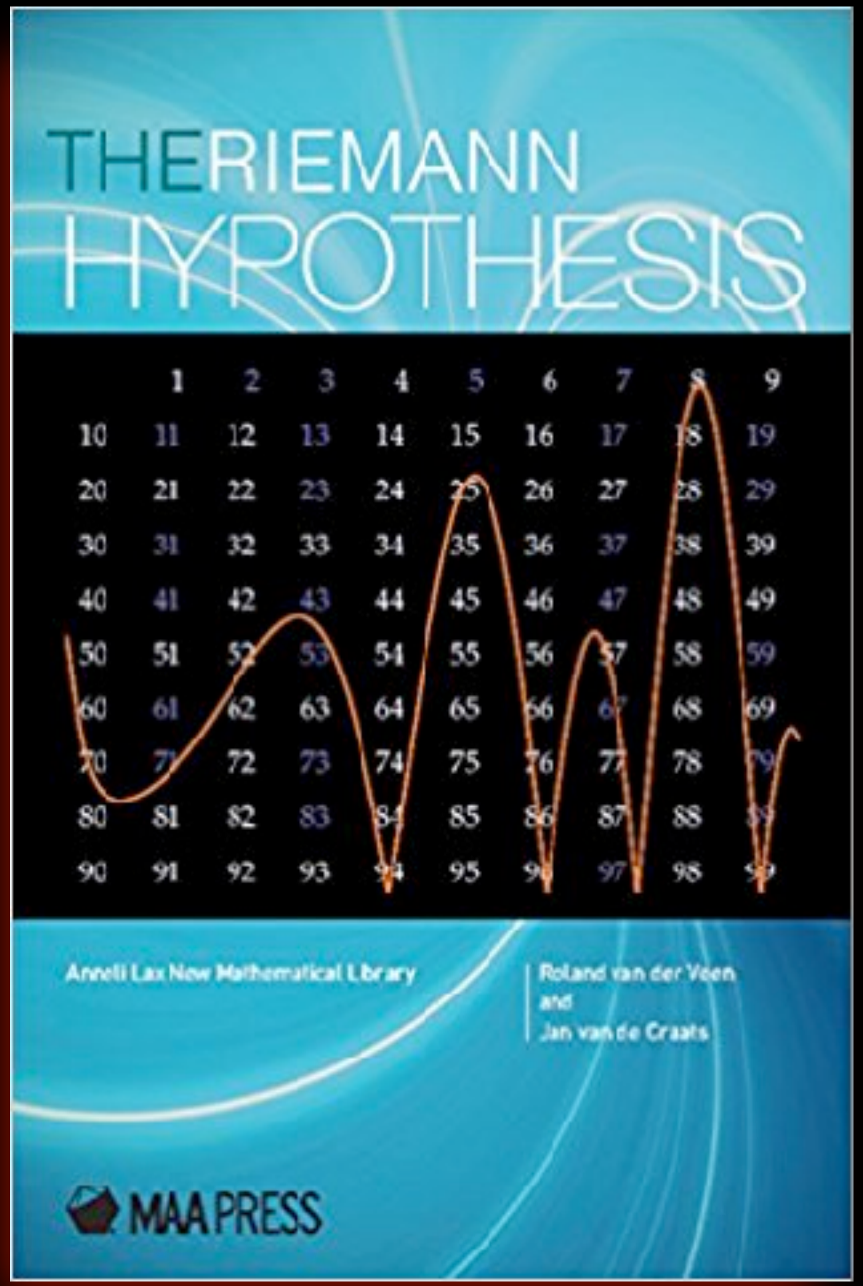


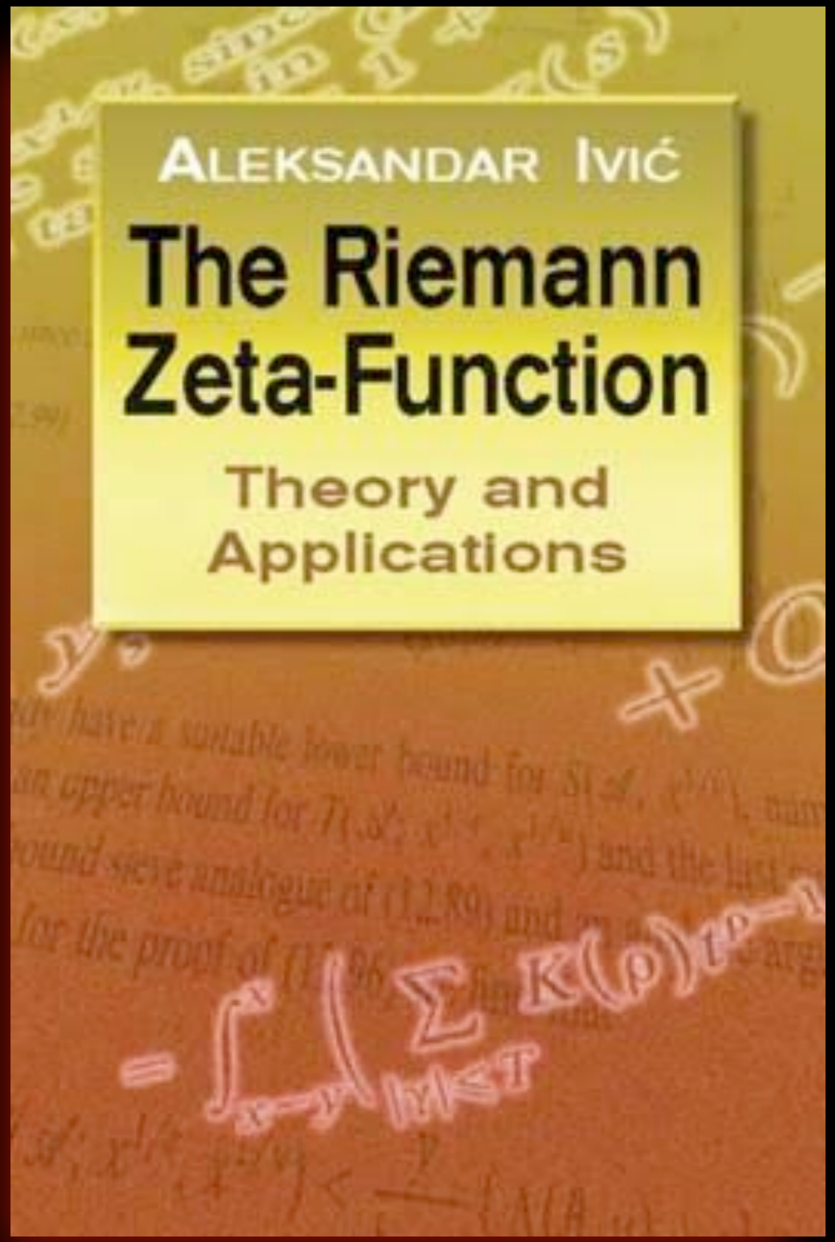



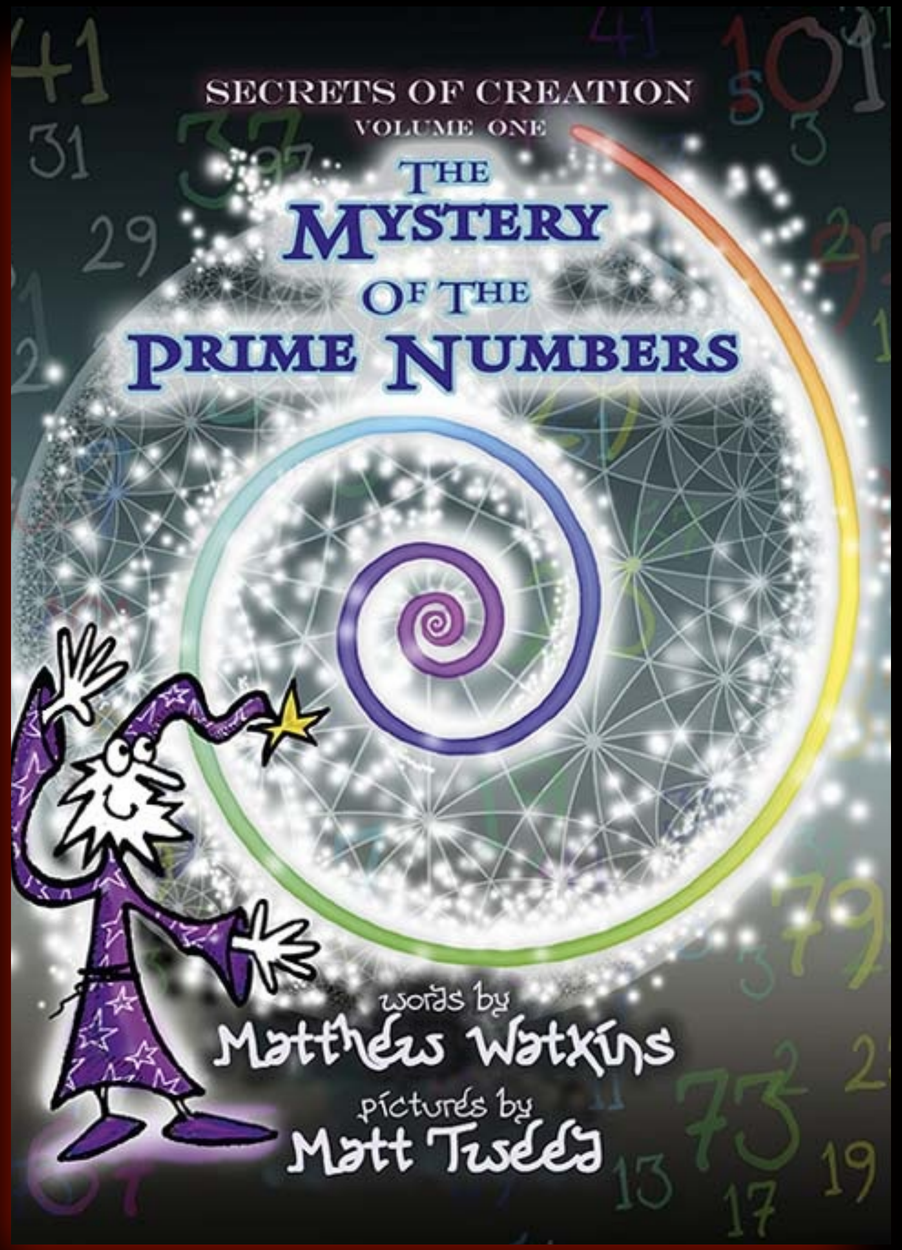

Prime Numbers and the Riemann Hypothesis
Barry Mazur
William Stein

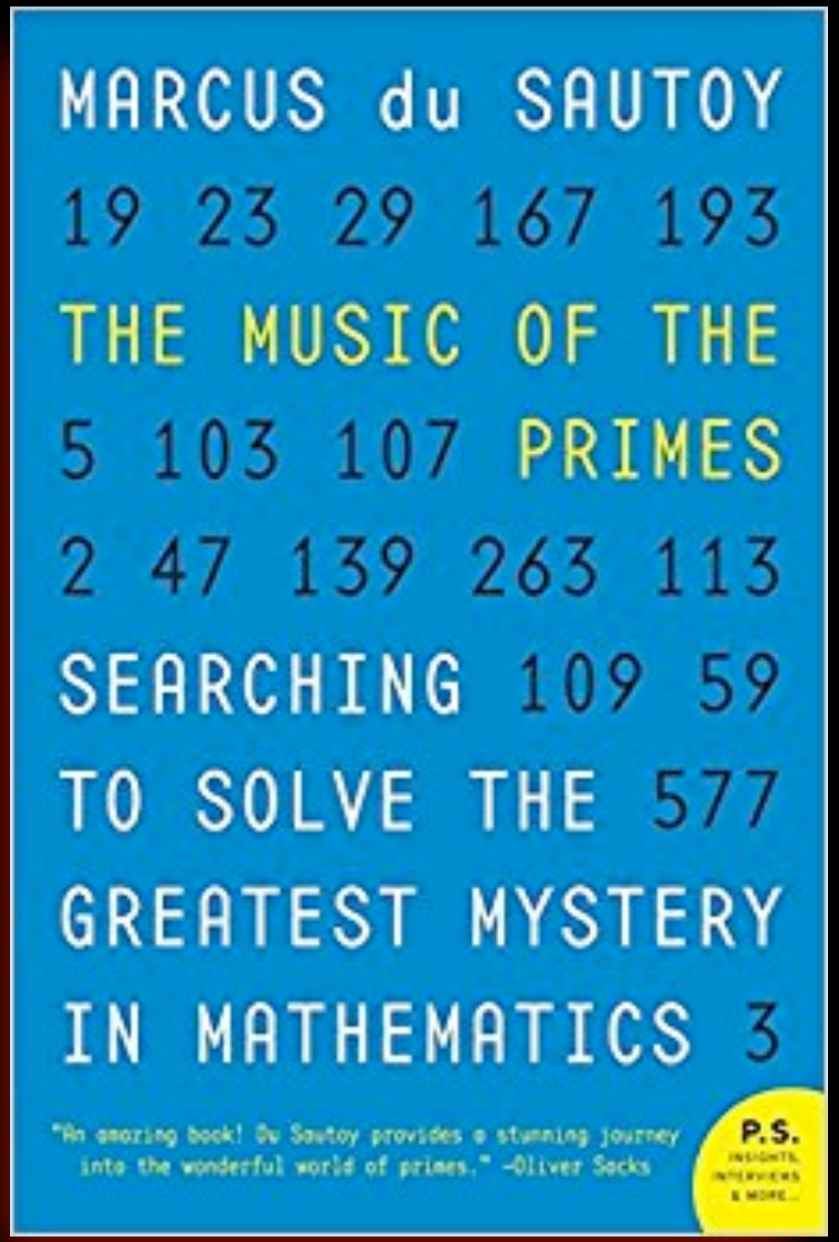

'A pleasurable and painless read for anyone intrigued by numbers'
Ian Stewart
Dr. RIEMANN'S ZEROS
THE SEARCH FOR THE $1 MILLION SOLUTION TO THE GREATEST PROBLEM IN MATHEMATICS
KARL SABBAGH

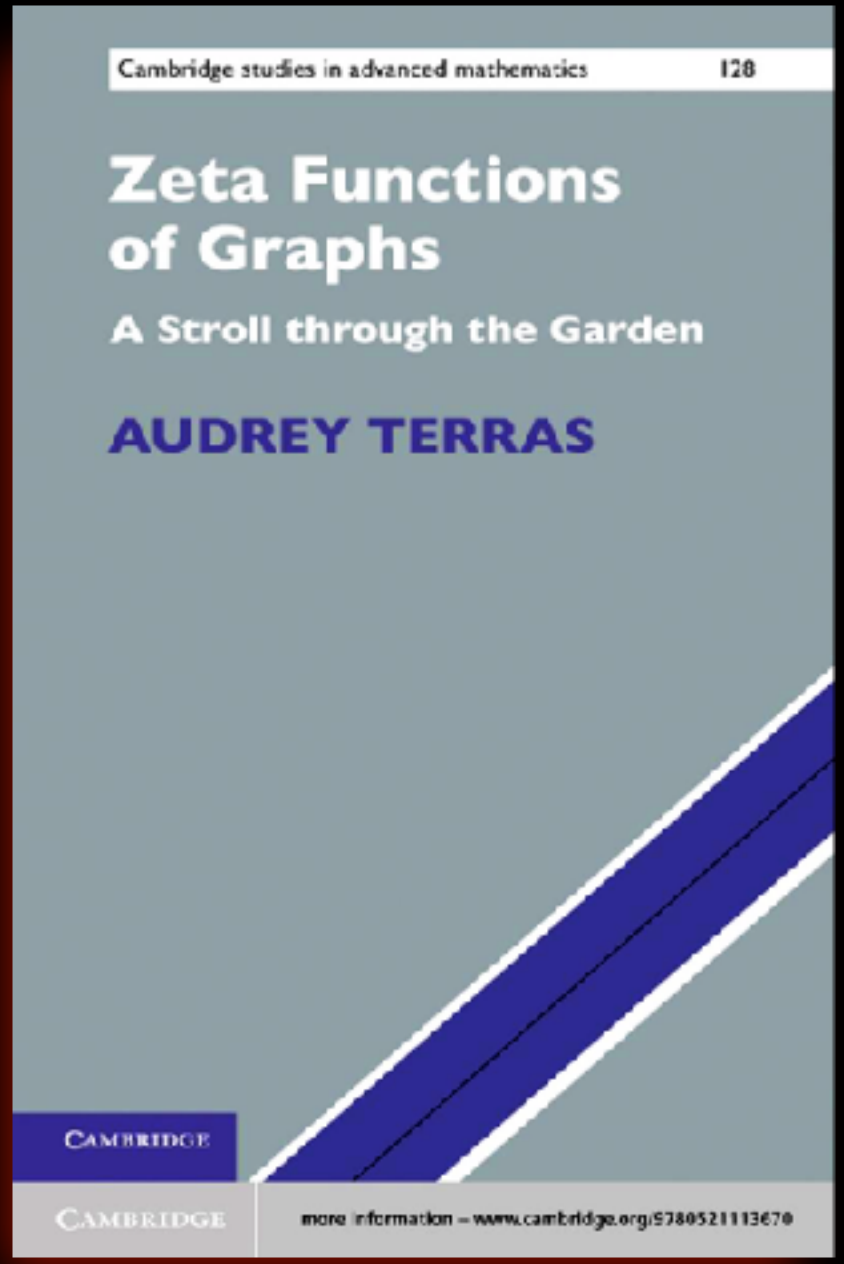

JOHN DERBYSHIRE
PRIME OBSESSION
BERNHARD RIEMANN AND THE GREATEST UNSOLVED PROBLEM IN MATHEMATICS


THE RIEMANN HYPOTHESIS
Anneli Lax New Mathematical Library
Roland van der Veen
and
Jan van de Craats
MAA PRESS

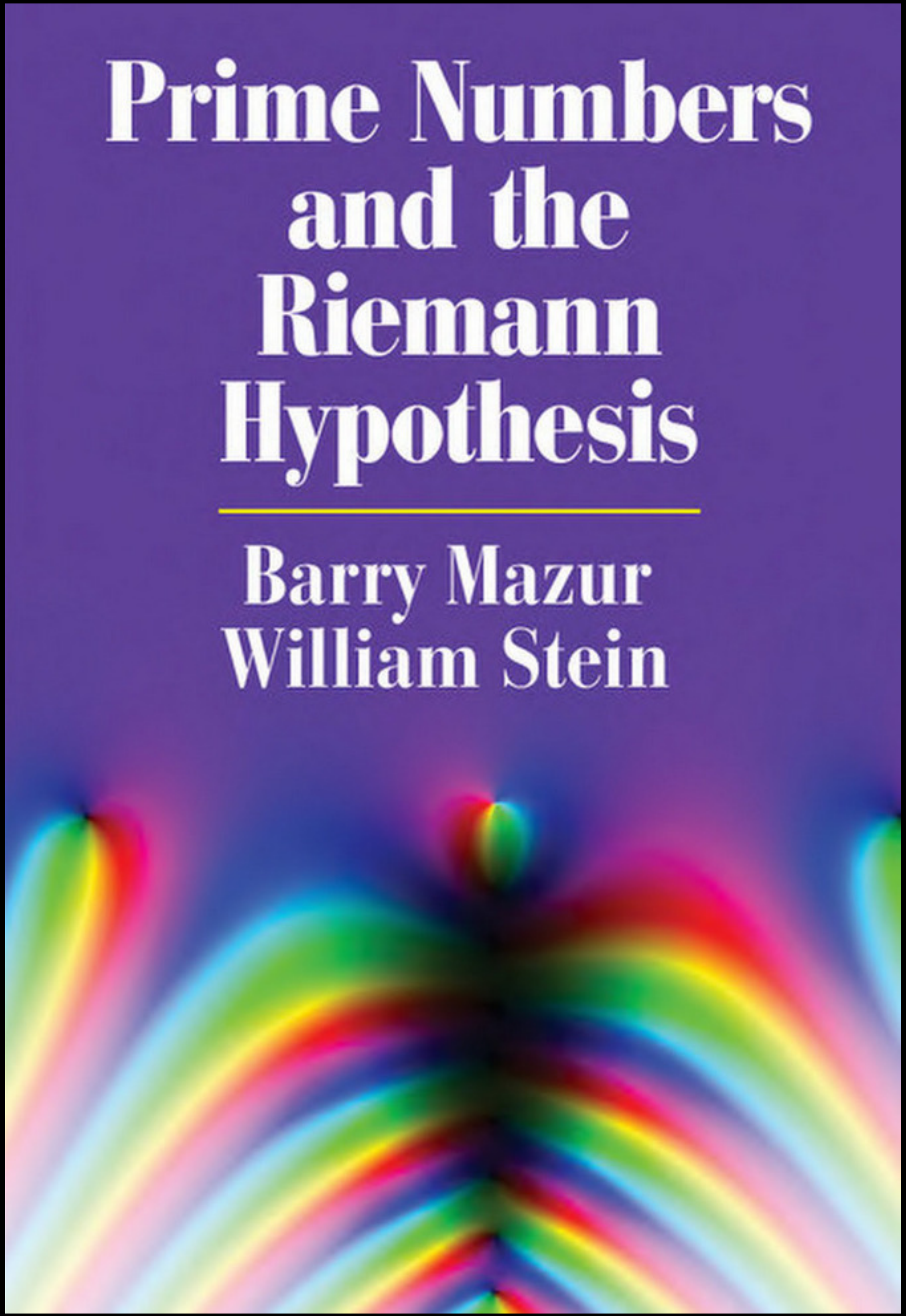

ALEKSANDAR IVIĆ
The Riemann Zeta-Function
Theory and Applications


SECRETS OF CREATION
VOLUME ONE
THE MYSTERY OF THE PRIME NUMBERS
words by
Matthew Watkins
pictures by
Matt Tweed


MARCUS du SAUTOY
19 23 29 167 193
THE MUSIC OF THE
5 103 107 PRIMES
2 47 139 263 113
SEARCHING 109 59
TO SOLVE THE 577
GREATEST MYSTERY
IN MATHEMATICS 3
"An amazing book! Du Sautoy provides a stunning journey into the wonderful world of primes." -Oliver Sacks
P.S.


Cambridge studies in advanced mathematics
128
Zeta Functions of Graphs
A Stroll through the Garden
AUDREY TERRAS
CAMBRIDGE

# 2016



Barry Mazur, William Stein

What is wonderful about this subject is that people attracted to it cannot resist asking questions that lead to interesting, and sometimes surprising numerical experiments. Moreover, given our current state of knowledge, many of the questions that come to mind are still unapproachable: we don't yet know enough about numbers to answer them. But *asking interesting questions* about the mathematics that you are studying is a high art, and is probably a necessary skill to acquire, in order to get the most enjoyment—and understanding—from mathematics. So, we offer this challenge to you:

Come up with your own question about primes that

- is interesting to you,
- is not a question whose answer is known to you,
- is not a question that you've seen before; or at least not exactly,
- is a question about which you can begin to make numerical investigations.

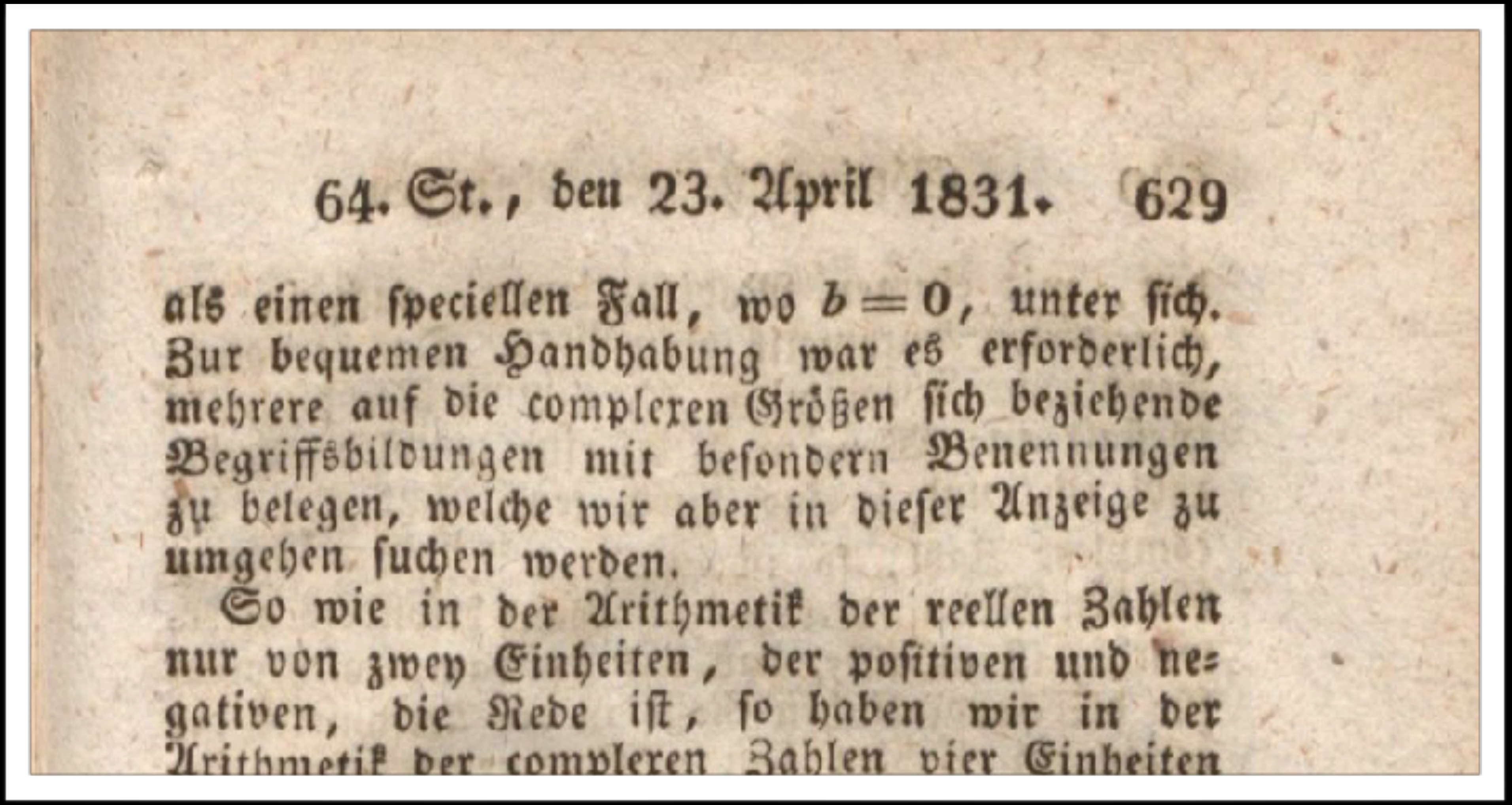
If you are having trouble coming up with a question, read on for more examples that provide further motivation.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 6 | 0 | 18 | 2 | 20 | 0 | 22 | 0 | 24 | 0 | 24 | 0 | 56 | 0 | 34 | 0 | 42 | 2 | 46 | 0 | 44 | 0 | 80 |
| 0 | 0 | 4 | 0 | 8 | 0 | 14 | 0 | 20 | 2 | 20 | 0 | 38 | 2 | 35 | 0 | 34 | 0 | 54 | 2 | 40 | 2 | 56 | 2 | 52 | 0 |
| 0 | 0 | 0 | 8 | 0 | 10 | 2 | 16 | 0 | 24 | 0 | 36 | 0 | 28 | 2 | 34 | 2 | 32 | 2 | 42 | 0 | 56 | 0 | 40 | 2 | 44 |
| 0 | 0 | 4 | 0 | 10 | 2 | 20 | 0 | 19 | 0 | 22 | 2 | 27 | 0 | 40 | 0 | 42 | 0 | 38 | 0 | 42 | 0 | 32 | 0 | 56 | 0 |
| 0 | 0 | 2 | 10 | 2 | 12 | 0 | 22 | 2 | 22 | 0 | 26 | 2 | 36 | 2 | 34 | 0 | 44 | 0 | 34 | 2 | 46 | 0 | 56 | 2 | 46 |
| 0 | 0 | 5 | 0 | 8 | 0 | 13 | 2 | 14 | 0 | 34 | 2 | 28 | 0 | 23 | 2 | 28 | 2 | 31 | 0 | 64 | 2 | 42 | 0 | 40 | 2 |
| 0 | 0 | 0 | 10 | 0 | 8 | 2 | 16 | 2 | 22 | 0 | 26 | 0 | 30 | 0 | 28 | 0 | 36 | 2 | 36 | 0 | 34 | 0 | 42 | 2 | 42 |
| 0 | 0 | 4 | 2 | 10 | 0 | 18 | 0 | 17 | 2 | 20 | 2 | 30 | 2 | 34 | 0 | 36 | 0 | 32 | 2 | 31 | 0 | 38 | 2 | 54 | 0 |
| 0 | 0 | 2 | 10 | 0 | 10 | 2 | 14 | 0 | 20 | 2 | 28 | 0 | 18 | 0 | 28 | 0 | 32 | 0 | 36 | 2 | 44 | 0 | 32 | 0 | 34 |
| 0 | 0 | 4 | 2 | 8 | 2 | 13 | 0 | 18 | 0 | 17 | 0 | 26 | 0 | 23 | 2 | 24 | 0 | 36 | 0 | 28 | 0 | 42 | 2 | 34 | 0 |
| 0 | 0 | 2 | 6 | 0 | 14 | 0 | 12 | 0 | 18 | 2 | 20 | 2 | 20 | 0 | 44 | 2 | 28 | 0 | 28 | 2 | 34 | 0 | 34 | 0 | 56 |
| 0 | 0 | 2 | 2 | 7 | 0 | 10 | 0 | 16 | 2 | 14 | 2 | 24 | 2 | 28 | 0 | 23 | 0 | 34 | 0 | 23 | 2 | 40 | 2 | 35 | 0 |
| 0 | 0 | 0 | 4 | 0 | 8 | 0 | 10 | 2 | 20 | 0 | 22 | 0 | 18 | 2 | 20 | 0 | 18 | 2 | 30 | 0 | 36 | 0 | 28 | 2 | 24 |
| 0 | 0 | 3 | 0 | 8 | 2 | 14 | 2 | 16 | 0 | 17 | 0 | 20 | 0 | 24 | 2 | 26 | 0 | 30 | 0 | 28 | 2 | 27 | 0 | 38 | 0 |
| 0 | 0 | 2 | 6 | 2 | 10 | 0 | 14 | 2 | 14 | 2 | 18 | 0 | 22 | 2 | 20 | 0 | 28 | 2 | 26 | 2 | 26 | 2 | 36 | 0 | 24 |
| 0 | 0 | 2 | 0 | 5 | 2 | 12 | 0 | 11 | 0 | 20 | 2 | 17 | 0 | 14 | 2 | 17 | 2 | 20 | 0 | 34 | 0 | 22 | 0 | 20 | 0 |
| 0 | 0 | 0 | 6 | 2 | 6 | 2 | 12 | 2 | 16 | 0 | 14 | 0 | 20 | 2 | 18 | 0 | 20 | 2 | 22 | 0 | 22 | 0 | 24 | 2 | 22 |
| 0 | 0 | 3 | 2 | 6 | 0 | 10 | 0 | 8 | 2 | 11 | 2 | 16 | 2 | 16 | 0 | 18 | 0 | 17 | 2 | 14 | 2 | 19 | 0 | 20 | 0 |
| 0 | 0 | 2 | 6 | 0 | 6 | 2 | 12 | 0 | 12 | 0 | 14 | 2 | 10 | 0 | 12 | 0 | 14 | 0 | 16 | 2 | 22 | 0 | 16 | 0 | 20 |
| 0 | 0 | 2 | 2 | 5 | 2 | 8 | 2 | 10 | 2 | 12 | 0 | 14 | 0 | 10 | 0 | 13 | 2 | 18 | 2 | 13 | 0 | 20 | 2 | 14 | 2 |
| 0 | 0 | 2 | 4 | 0 | 4 | 2 | 6 | 0 | 6 | 2 | 10 | 2 | 8 | 0 | 14 | 2 | 10 | 0 | 8 | 0 | 12 | 2 | 10 | 0 | 18 |
| 0 | 0 | 1 | 2 | 4 | 0 | 5 | 0 | 6 | 2 | 5 | 2 | 8 | 0 | 7 | 0 | 8 | 0 | 10 | 0 | 8 | 2 | 10 | 0 | 8 | 0 |
| 0 | 0 | 2 | 2 | 2 | 4 | 2 | 6 | 2 | 6 | 0 | 6 | 0 | 4 | 2 | 6 | 2 | 10 | 2 | 10 | 0 | 10 | 0 | 8 | 0 | 6 |
| 0 | 0 | 1 | 2 | 1 | 2 | 2 | 2 | 3 | 0 | 2 | 2 | 3 | 0 | 2 | 2 | 4 | 2 | 4 | 0 | 5 | 2 | 4 | 0 | 4 | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# *2016*

Every even Gaussian integer a+ib with a>1,b>1 is a sum of two positive Gaussian primes

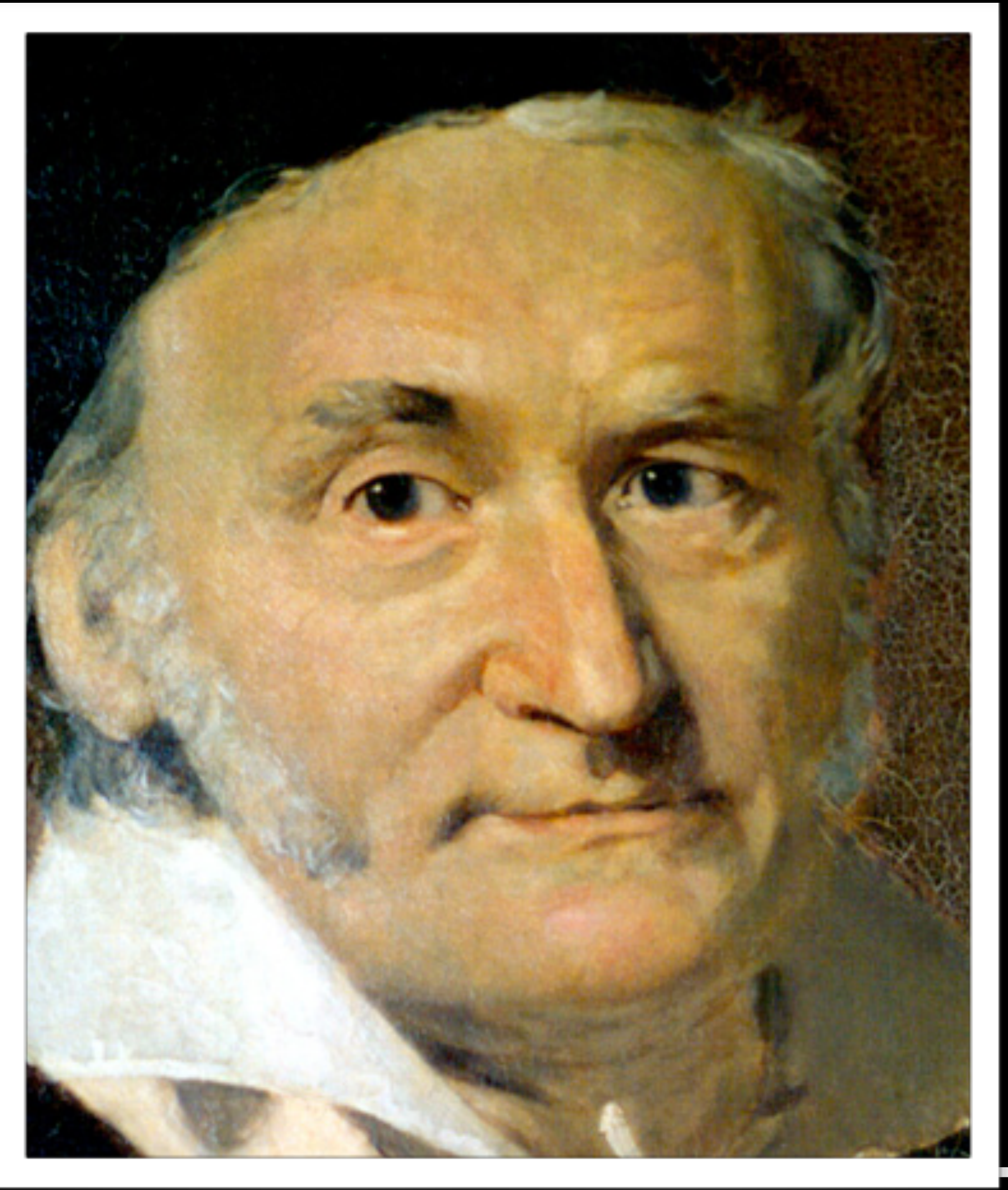
# *Gaussian Integers*

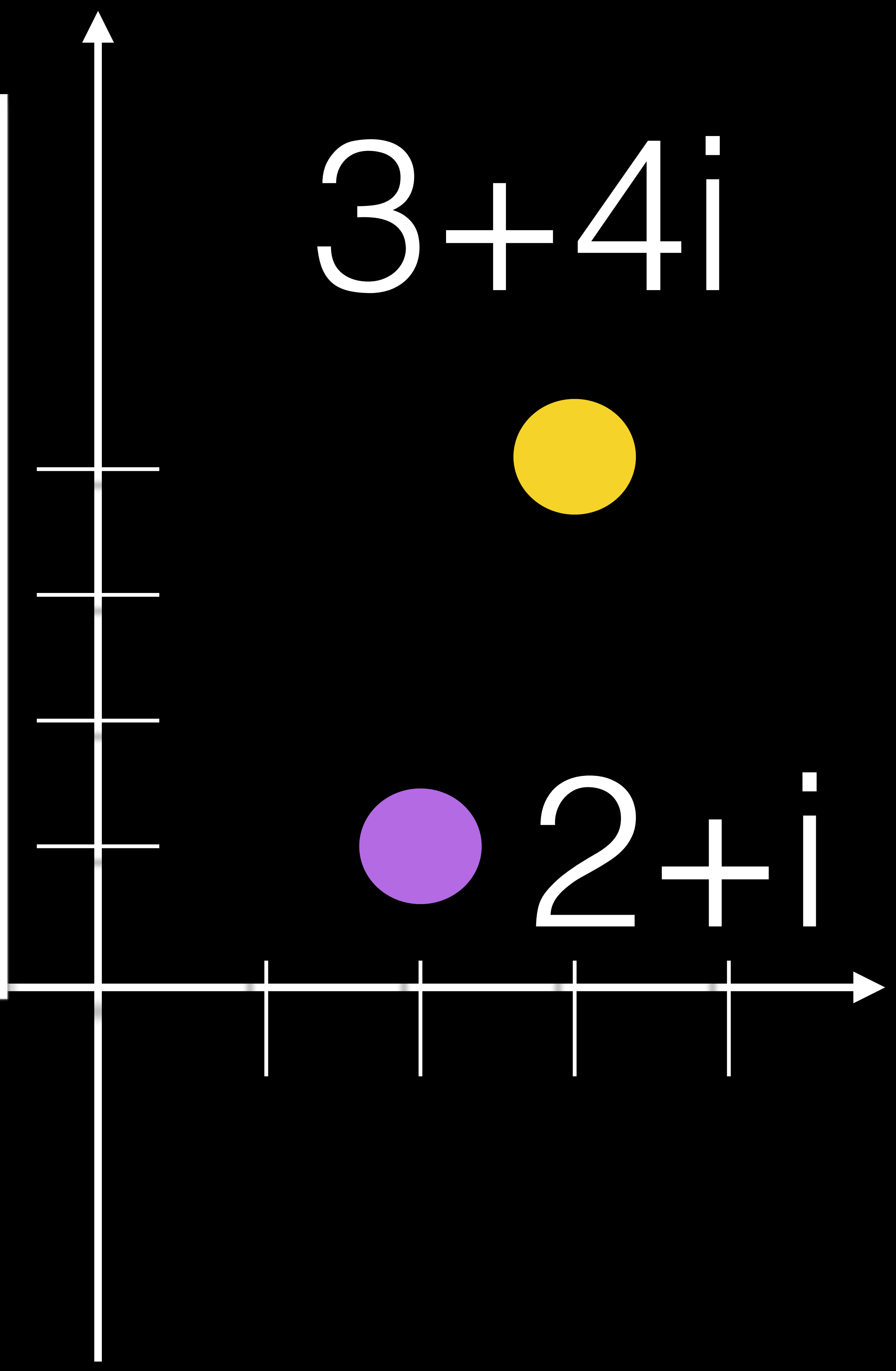


64. St., den 23. April 1831. 629

als einen speciellen Fall, wo $b = 0$, unter sich. Zur bequemen Handhabung war es erforderlich, mehrere auf die complexen Größen sich beziehende Begriffsbildungen mit besondern Benennungen zu belegen, welche wir aber in dieser Anzeige zu umgehen suchen werden.

So wie in der Arithmetik der reellen Zahlen nur von zwey Einheiten, der positiven und negativen, die Rede ist, so haben wir in der Arithmetik der complexen Zahlen vier Einheiten






$$5=(2+i)(2-i)$$

$$3+4i=(2+i)(2+i)$$

are not Gaussian primes

$2+i$ is Gaussian prime

# *Gaussian Comet*

$$f(x,y) = x\,y + x^{2}y + x^{4}y + x^{6}y + x^{10}y + \ldots$$

$$g(x,y) = x^{2}y^{2} + x^{2}y + x^{4}y + x\,y^{2} + \ldots$$

```
n = 20;
f =  Sum[If[PrimeQ[a + I b, GaussianIntegers -> True], x^a y^b, 0], {a,n}, {b, n}];
g = Expand[f*f]; G = CoefficientList[Expand[f*f], {x, y}];
```


Gaussian primes
Im
3i
Re
3
7
units

*The End*