

GOLDBACH COMETS

OLIVER KNILL, MATH TABLE, OCTOBER 23, 2018

1. THE LANDAU PROBLEMS

1.1. Edmund Landau nailed it! In 1912, he listed four problems on prime numbers. All conjectures remain unsolved. The problems are elementary and each can be formulated in 9 words! Here are the now famous **Landau conjectures**:

Goldbach: Every integer $2n > 2$ is a sum of two primes.

Twin prime: There are infinitely many primes $p, p + 2$, prime twins.

Legendre: Between two consecutive perfect squares, there is a prime.

Landau: There are infinitely many primes of the form $n^2 + 1$.

2. THE COMET

2.1. To compute the comet, look at $f(x) = \sum_{p \in P} x^p = x^2 + x^3 + x^5 + x^7 + \dots$, where p runs over the set of rational primes $P = \{2, 3, 5, 7, 11, \dots\}$, then form

$$g(x) = f(x)^2 = x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + 2x^{12} + 2x^{13} + 3x^{14} + \dots .$$

It tells for example that $n = 10$ can be written in three ways as a sum of two primes. Indeed $5 + 5$ or $3 + 7$ or $7 + 3$.

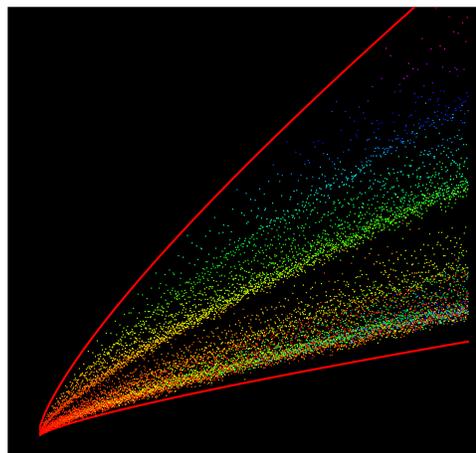


FIGURE 1. The Goldbach comet and the suspected lower and upper bounds which are of the form $Cn/\log(n)^2$. We explain the bound in the talk.

3. MODIFYING THE COMET

3.1. We do not need the coefficients of f to be 1. A **golden bullet** already would be to find a sequence a_n of positive numbers for which $f(x) = \sum_p a_p x^p$ can be expressed using **known functions**. We can then look at the Taylor expansion of $f(x)^2$ and hope to prove that all $2n$ 'th derivatives are positive, establishing so Goldbach. Nobody has proven that such a function does not exist, so that this is a perfect topic for **experimental mathematics**

3.2. Remembering the exponential function $e^x = 1 + x/1! + x^2/2! + \dots$ motivates to look at

$$f(x) = \sum_{p \in P} \frac{x^p}{p!} = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots .$$

Now, the coefficients b_n of $g(x) = f(x)^2$ are rational. They go to zero very quickly and it is better to look at $\log(b_n)$. It is a bit of a surprise but they are much more regular:

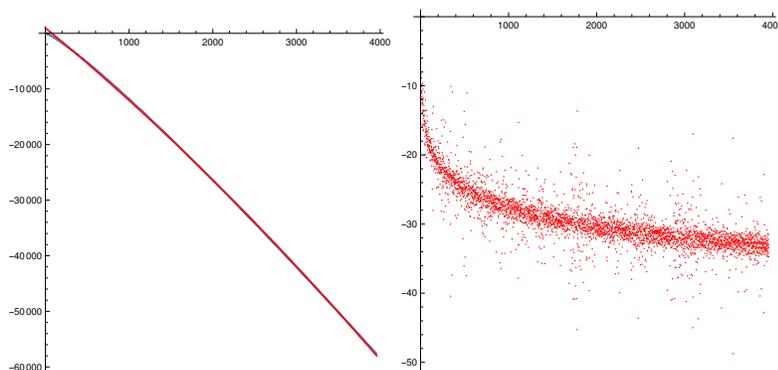


FIGURE 2. The Goldbach comet of the factorial prime function $f(x) = \sum_{p \in P} \frac{x^p}{p!}$ is more regular. Again, establishing a lower bound for this function would do the job. The second graph shows the difference. As it is negative, it suggests monotonicity.

3.3. Let b_n be the coefficients for f^2 if $f = \sum_p \frac{x^p}{p!}$.

Observation: $b_{n+4} - b_n \leq 0$.

3.4. This observation would establish the Goldbach conjecture as no coefficient could be zero. Now, one has to be careful. We deal here with extremely small numbers and numerics could fool us. Still, as the coefficients, the logarithms of rational numbers are evaluated rather well, also for rational function p/q with extremely large integers p, q . What this experiment suggests however is that Goldbach is very robust. We can experiment with random sequences P , which have the same density than primes and it looks as if one survives the first part, where by chance some coefficients can be zero, the large enough coefficients are always positive.

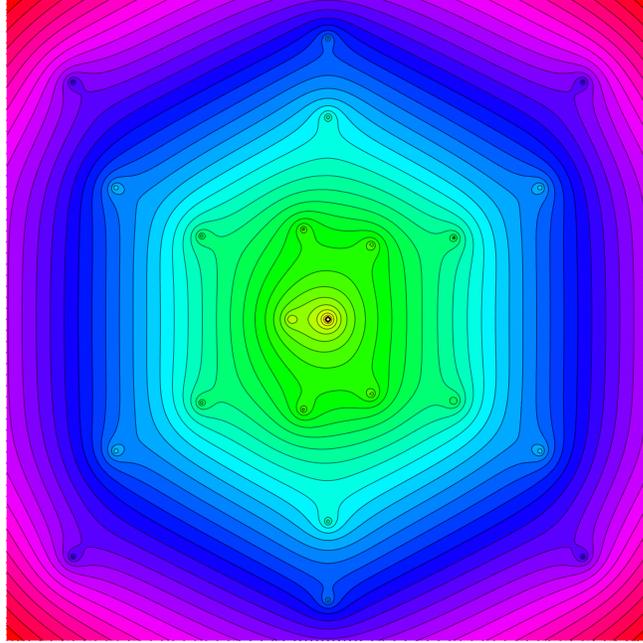


FIGURE 3. The contour of the absolute value of the complex function $f(z)^2$, an entire function. We can compute the derivatives using the Cauchy integral formula $b_k = \frac{1}{2\pi i} \int_C \frac{g(z)}{z^{k+1}} dz$, where γ is a circle.

4. COMPLEX SITUATION

4.1. The Hardy-Littlewood circle method uses complex analysis. We can look at $g(z) = f(z)^2$ for complex z and plot the level curves of the absolute value.

5. EULER GOLDEN KEY

5.1. As we have a sum of primes, lets look at the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The first equality is a definition, the second equality is the **Euler golden key**. It suggests to look at

$$f(x) = - \sum_p \log\left(1 - \frac{1}{p^s}\right) x^p.$$

As we want to stay in real analysis, we assume $s > 1$ is real. This Goldbach function has the property that $f(1) = \log(\zeta(s))$. It is an analytic function in x for $|x| < 1$. Again, in complex analysis we would compute the derivatives using **contour integrals**. It goes back to the **Hardy-Littlewood circle** method.

6. EXPERIMENTS WITH FOURIER

6.1. One can also try to stay in the real and make a **Fourier series** $\sum_{k=1}^{\infty} a_k \sin(\pi k x)$. Now, using the trigonometric identity $2 \sin(X) \sin(Y) = \cos(X - Y) - \cos(X + Y)$ we have $f(x)^2 = \sum_{k,l} a_k a_l [\cos(\pi(k - l)x) - \cos(\pi(k + l)x)]/2$. Now the $2n$ 'th derivative is

$\sum_{k,l} a_k a_l \pi^{2n} A_{n,k,l} / 2$, where $A_{n,k,l} = -(k-l)^{2n} + (k+l)^{2n}$. We observe that $a_k \sim 1/\sqrt{k}$ for even k and $a_k \sim -1/\sqrt{k}$ for odd k .

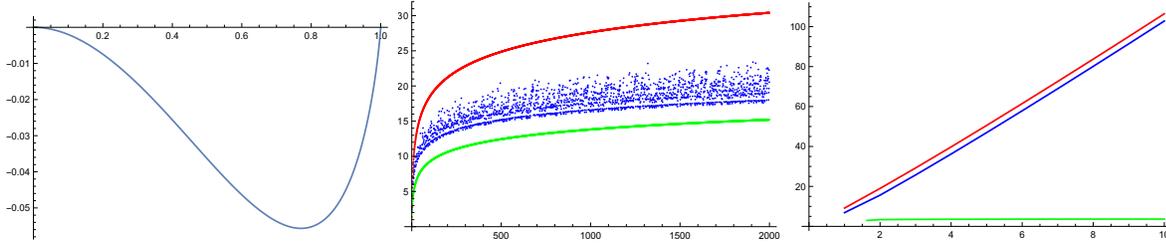


FIGURE 4. The Zeta Goldbach function $f(x)^2 - \zeta(2)x^2$ in the case $s = 2$. We see also the corresponding Goldbach comet which appears to grow like $\log(n)$. The third picture shows the coefficients a_k in the case of $c_p = -\log(1 - 1/p^2)$.

7. THE CASE OF GAUSSIAN INTEGERS

7.1. If $Q = \{a + ib \mid a > 0, b > 0\}$ is the first quadrant in \mathbb{C} , one can ask whether an even Gaussian integer in Q can be written as a sum of two Gaussian primes in Q . This has a multi-variable calculus reformulation: define

$$f(x, y) = \sum_{p=a+ib \in Q} x^a y^b,$$

where the sum is over all Gaussian primes in Q . Now $f(n, m)^2$ has coefficients which tell how many times one can write a $n + im$ as a sum of two Gaussian primes in Q . The Gaussian Goldbach conjecture states that every even Gaussian integer in Q can be written as a sum of two Gaussian primes in Q . (A Gaussian integer is **even** if it is divisible by $1 + i$.)

7.2. We can try the same zeta approach as Gaussian integers have a zeta function with a Golden key formula

$$\sum_{n \in Q} \frac{1}{N(n)^s} = \prod_{p \in Q} \left(1 - \frac{1}{N(p)^s}\right)^{-1},$$

where $N(a + ib) = a^2 + b^2$ is the arithmetic norm. Chances for a success are slimmer even because the Gaussian Goldbach conjecture implies the Landau conjecture, which currently appears to be hopeless!

7.3. There are also interesting Goldbach conjectures for Eisenstein integers, for Quaternions and even for Octonions, so that all normed division algebras are covered: $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$.