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Intriguing open problems in Hamiltonian dynamics

This is a personal list of problems. Statements after GUESS: answer chosen when I would be asked for an opinion. Statements after SOURCE: starting points in the literature.

UNSTABLE EXTERIOR BILLIARDS. Is there a convex table in the plane such that the exterior billiard map T at this table has an unstable orbit ($|T^n(x,y)|$ unbounded).

KNOWN: Stability for smooth tables by KAM and classes of polygons (Vivaldi-Shaidenko).
GUESS: All convex tables are stable. If this is wrong and a table with an unstable example should exist, the table should be obtained by a Baire category argument.
SOURCE: Problem posed by Hammer in 1960. See J. Moser "Stable and random Motion in dynamical systems", 1973, J. Moser, "Is the solar system stable", 1978., Tabachnikov, "Billiards", 1995

N-BODY PROBLEM: What is the measure of the set of initial conditions of the Newtonian n-body problem leading to global solutions? The complement, the set of **singularities** splits into collision and non-collision singularities.

KNOWN: Collision singularities have measure zero (Saari). Non-collision singularities exist for $N \geq 5$ particles (Xia). In 4 dimensions, collisions can have positive measure. Almost everywhere existence for N=3,4 particles known (Saari). Smooth mass distributions (Vlasov generalization of n-body problem) leads to global solutions (Pfaffelmoser 1992).
GUESS: can an almost everywhere existence result be obtained by extending the Vlasov existence theorem to a BBGKY hierarchy?
SOURCE: Probably first posed by Painlevé or Poincaré. The problem appears in the list of B.Simon: "Fifteen problems in mathematical physics", 1984 as Problem 1A. For Vlasov, see R.T. Glassey "The Cauchy problem in kinetic theory".

BIRKHOFF BILLIARDS: Is there a smooth Birkhoff billiard (a billiard in convex table) for which the dynamics has positive entropy? The entropy is by Pesin given by the averaged Lyapunov exponent $\limsup_{n\to\infty} \int_X n^{-1} \log ||dT^n(s,u)|| \; dsdu$, where dT^n is the Jacobean of the n'th iterate of the billiard map T.

KNOWN: Non-smooth examples like the Bunimovich stadium are known. Smooth, strictly convex billiards are not ergodic. (KAM leads to "whisper-galleries" near the boundary).
GUESS: Most smooth Birkhoff billiards should have positive entropy. Positive entropy measured for example for the table $x^4 + y^4 = 1$.
SOURCE: Problem probably first posed by Y. Sinai. See S. Tabachnikov, "Billiards", 1995. For entropy measurements, see for example M. Jeng and O. Knill, "Billiards in the l^p unit balls of the plane", 1996

BIRKHOFF-PORITSKI CONJECTURE. Is every integrable Birkhoff billiard an ellipse? Related question: if a Birkhoff billiard map T is conjugated to an algebraic map, is the table then an ellipse.

KNOWN: Problem depends on the definition of integrability. One possible definition: every ergodic invariant measure leads to a dynamical system with discrete spectrum. Under a strong integrability condition (foliation of the phase space by invariant circles), the table must be a circle (Bialy). A real-analytic table with integral which is analytic in the angle must be an ellipse.
GUESS: There might be counter examples of tables of constant width (de Verdière) and these billiard maps are non-algebraic.
SOURCE: Problem posed by H. Poritsky, Annals of Math, 51, 1950. Unclear whether it is a question of Birkhoff.

FRACTAL CAUSTICS.
Are there geodesic flows or Birkhoff billiards with fractal caustics?

KNOWN: Examples of tables with nowhere differentiable caustics are known (tables of equal width).
GUESS: For any in $s \in [1,2)$, there is a caustic of a convex billiard with Hausdorff dimension s. For any $s \geq 1$ there exists a Riemannian manifold M and a point such that the caustic of this point has Hausdorff dimension s.
SOURCE: O. Knill, "On non-convex caustics of convex billiards", 1998

CONJUGACY. If T_1, T_2 are smooth Birkhoff billiard maps. If $T_1 = ST_1S^{-1}$ with a homeomorphism S, then the tables are similar. A related question is the Kac inverse problem: "Can one hear the shape of a **convex** drum?"

KNOWN: The shape of non-convex tables can not be heard because there are isospectral nonconvex tables (Gordon-Webb following Buser,Sundada)
GUESS: One should be able to hear the shape of a real-analytic convex drum. Conjugacy of two Birkhoff billiard maps should imply similarity of such tables.
SOURCE: The billiard conjugacy problem was posed by V.Guillemin in the late seventies.

PERIODIC ORBITS. a) The set of n-periodic orbits of a smooth strictly convex Birkhoff billiard is nowhere dense for all n. b) Does every polygonal Birkhoff billiard have a periodic orbit?

KNOWN: Yes, for $n = 2, 3, 4$, periodic orbits are nowhere dense (Rychlik). Polygons with angles which are a rational multiply of π have periodic orbits (actually a generic orbit is periodic (Boshernitzan-Stepin)).
GUESS: a) All n-periodic orbits are nowhere dense. b) All polygonal Birkhoff billiards should have periodic orbits.
SOURCE: S. Tabachnikov, "Billiards", 1995

FREE GAS. Does a free gas coupled to a convex rigid container by the law of momentum conservation converge weakly to an equilibrium and the container (which only moves by translation) come to rest?

KNOWN. The dynamics exists. For a discrete mass distribution, the Hamiltonian system is finite dimensional and there is Poincaré recurrence.
GUESS. For initial gas and phase space densities near equilibrium in one dimensions the boundary comes to rest like $1/t$. In general, it should depend on the dynamical properties of the billiard obtained when the container is at rest. i.e. exponential convergence with strong mixing properties of the billiard would imply exponential fast convergence to equilibrium.
SOURCE. O. Knill, "An existence theorem for Vlasov dynamics with moving boundaries. MP-ARC document 00-38

KOLMOGOROV PROBLEM. Is there a Hamiltonian system with a smooth invariant torus, on which the induced dynamics is mixing?

KNOWN: Situations with weakly mixing tori. No mixing can happen on two-dimensional tori.
GUESS. Yes, there should exist examples in higher (d¿3) dimensions.
SOURCE. According to Arnold, this is a question of Kolmogorov which motivated KAM theory. (See O. Knill, Weakly mixing invariant tori of Hamiltonian systems, 1999)

THE GOOD THE BAD AND THE UGLY. Call the "good set" the maximal invariant subset in the phase space of a Hamiltonian systems for which the invariant Liouville measure is almost periodic. Call the "bad set" the Pesin set on which the invariant measure has a positive Lyapunov exponent. Call the "ugly set" the complement of the union of these two sets. Is there an example where the ugly set has positive measure (that is, where it is physically relevant)?

KNOWN. The GOOD SET has often positive measure (KAM). The bad set is believed to have positive measure in many cases.
GUESS: The ugly set has positive measure too in general and the dynamics on this set has some weak mixing properties on sets of positive measure.
SOURCE: O. Knill, Singular continuous spectrum and quantitative rates of weakly mixing, 1998

MANÉ'S LAST THEOREM: In the set of area-preserving C^1 diffeomorphisms on a compact manifold, one has generically hyperbolicity or zero Lyapunov exponents?

KNOWN. The hyperbolic diffeomorphisms form an open set. Away from diffeomorphisms with hyperbolicity, one generically has a dense set of elliptic orbits (Newhouse). Yes, for dim=2. (preprint of J. Bochi, Sept. 15, 2000).
GUESS. Away from Anosov maps, there exists a dense set of Newhouse maps: diffeomorphis having a set of periodic orbits with full measure. This would imply Mané's last theorem and also show that generically, an area preserving non-Anosov C^1 map has a Koopman operator with singular continuous spectrum.
SOURCE. R. Mané, "The Lyapunov exponents of a generic area-preserving diffeomorphism, 1995".

CALOGERO-MOSER-VLASOV. The Calogero-Moser system is a n-body problem on the real line, where particles interact with the inverse square potential. The question is whether in infinite-dimensions the dynamics exists. As for any infinite-dimensional integrable system, the question arrizes whether it is integrable in the sense that every ergodic invariant measure gives rize to almost periodic dynamics.

KNOWN. There is an existence proof for some continuous measures located on one-dimensional sets (at each point a particle has a specific speed).
GUESS. Yes, the dynamics should exist for any smooth initial measure in the phase space and the motion is conjugated to an isospectral deformation of a Calderon-Zygmund operator.
SOURCE. J. Moser, "Three Integrable Hamiltonian Systems Connected with Isospectral Deformations".

MATHER THEORY NEAR INTEGRABLE SYSTEMS. Are there quasiperiodic global minimals for metrics on the torus which are close to a flat three dimensional torus? (A geodesic is a global minimal if for any two points on the geodesic, the piece between them is a minimal solution.)

KNOWN. Not necessarily for metrics far away from integrable situations (Hedlund examples). Yes in two-dimensions (Mather theory).
GUESS: Beside the KAM transition, there should be a transition, where Mather theory breaks down.
SOURCE: "J. Moser, Selected topics in the calculus of variations, ETH, 1988".