4. Talk: Homogenisation, positive Lyapunov exponents for new classes of cocycles

UPPER AND LOWER FLUCTUATION FUNCTIONS

$f = \int_{\mathbf{C}} \log|z-w|\; dk(w)$ — subharmonic function
$M_f(z) = \sup_{|w|=|z|} f(w)$ — maximal function
$m_f(z) = \inf_{|w|=|z|} f(w)$ — minimal function
$C_f(z) = \int_0^{2\pi} f(\exp(2\pi i\theta)z)\; d\theta$ — mean function
$B_f(z) := M_f(z) - C_f(z) \geq 0$ — upper fluctuation
$A_f(z) := C_f(z) - m_f(z) \geq 0$ — lower fluctuation

REMARKS
M_f subharmonic (Hadamard three circle theorem).
C_f subharmonic
M_f and so B_f bounded
m_f and so A_f can be unbounded:
Example: $f(z) = \log|z-1|$.

RIESZ MEASURE AND RADIAL MEASURE
$dk_f = (2\pi)^{-1}\Delta f$ — Riesz measure
$d\rho_f(r) = \int_0^1 dk(re^{2\pi i\theta})\; d\theta$ — Radial measure

HOMOGENISATION TRANSFORM. ρ finite Borel probability measure on $\mathbf{R}^+$. Define $\mathcal{K}_l(\rho)(r) = 2^{-l}\sum_{k=l}^\infty \int_{\mathbf{R}^+} \exp(-|\log(r/t)|2^k)\; d\rho(t)$ and $\mathcal{K} = \mathcal{K}_0$.

HOMOGENISATION LEMMA.
$A_f(z) \leq \sum_{k=0}^\infty B_{f^{(k)}}(z)$.
$f(z) = f(\xi_l z) \Rightarrow A_f(z) \leq 2^{-l}\sum_{k=l}^\infty B_{f^{(k)}}(z)$

k'TH HOMOGENIZED FUNCTION
$\xi_k = \exp(2\pi i/2^k)$ 2^k'th primitive root of unity.
$f^{(k)}(z) = \sum_{j=1}^{2^k} f(z\xi_k^j)$

PROOF. Define $\xi_{k,l} = \xi_k\xi_{k-1}\cdots\xi_l$. Recursively apply $f^{(k)}(z) = f^{(k-1)}(z) + f^{(k-1)}(\xi_k z)$

$$
\begin{aligned}
f^{(k)}(z) &= f^{(k-1)}(z) + f^{(k-1)}(\xi_k z) \\
&= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-2)}(\xi_{k,k-1} z) \\
&= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-3)}(\xi_{k,k-1} z) + f^{(k-3)}(\xi_{k,k-2} z) \\
&\cdots \\
&= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + \cdots + f^{(1)}(\xi_{k,3} z) + f^{(0)}(\xi_{k,2} z) + f^{(0)}(\xi_{k,1} z)
\end{aligned}
$$

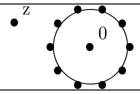
So, $f^{(0)}(\xi_{k,1}z) = f^{(k)}(z) - \sum_{j=0}^k f^{(j)}(\xi_{k,j+1}\xi_m z)$. Subtract the angular means $f(\xi_{k,1}z) - C_f(z) = f^{(k)}(z) - 2^k C_f(z) - \sum_{j=0}^k (f^{(j)}(\xi_{k,j+1}z) - 2^j C_f(z)) \geq (f^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^k (M_{f^{(j)}}(z) - C_{f^{(j)}}(z))$. Apply this to subharmonic functions $f_n(z) = 2^{n-1}\int_{-2^{-n}}^{2^{-n}} f(ze^{ix})\, dx$, for which $M_{f_n^{(k)}}(z) \leq M_{f^{(k)}}(z), C_{f_n} = C_f$, $B_{f_n^{(k)}} \leq B_{f^{(k)}}$. Get $f_n(\xi_{k,1}z) - C_f(z) \geq (f_n^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^k B_{f^{(j)}}(z)$. In limit $k \to \infty$ using $(f_n^{(k)}(z) - 2^k C_f(z)) \to 0$ for $k \to \infty$ gives $f_n(z) - C_{f_n}(z)) \geq -\sum_{j=0}^\infty B_{f^{(j)}}(z)$. n-independence of RHS $\Rightarrow$ $\inf_n(f_n(z) - C_f(z)) \geq -\sum_{j=0}^\infty B_{f^{(j)}}(z)$. Have $\inf_{n>0} f_n(z) \leq f(z)$ by upper semicontinuity. Therefore $f(z) - C_f(z) \geq -\sum_{j=0}^\infty B_{f^{(j)}}(z)$ so that $f(z) - C_f(z) \geq -\sum_{j=0}^\infty B_{f^{(j)}}(z)$. If $f(z) = f(\xi_l z)$, start with $2^l f(\xi_{k,1}z) = f^{(k)}(z) - \sum_{j=l}^k f^{(j)}(\xi_{k,j+1}\xi_m z)$.

HOMOGENISATION THEOREM.

$$A_f(z) \leq \mathcal{K}(\rho_f)(|z|)$$

$$f(z) = f(\xi_l z) \Rightarrow A_f(z) \leq \mathcal{K}_l(\rho_f)(|z|).$$

CIRCULAR CHAIN of charges: $f(z) = \log|z^m - a|$
$B_f(z) \leq \exp(-|\log(|z|/|a|)|m)$



PROOF OF THE HOMOGENISATION THEOREM.
If $f(z) = \log|z-a| \Rightarrow B_{f^{(k)}}(z) \leq \exp(-|\log(r/|z|)|2^k)$.
Proof. $\log(1+x) \leq x$, $\forall x \geq 0$ implies $\log|1+(|z|/r)^{2^k}| \leq (|z|/r)^{2^k} = \exp(-\log(r/|z|))$ for $|z| > r$ and $\log|1+r/|z|)^{2^k}| \leq (r/|z|)^{2^k} = \exp(-|\log(r/|z|)|)$ for $|z| < r$. Apply potential for circular chain.
If $f(z) = \int_{\mathbf{C}} \log|z-w|\; dk(w)$, the estimate is obtained by integration.

APPLICATION. f general subharmonic function. $\delta > 0$, $z \in \mathbf{C}$ with $f(z) > -\infty$. Then $Y = \{w \mid f(w) > f(z) - \delta\}$ has full Lebesgue density at z. Almost all sequences $z_n \to z$ have property that $f(z_n) \to f(z)$. "Subharmonic functions are almost continuous".

$\mathcal{K}_1$ FINITE ON L^p. $(1/p + 1/q = 1)$ $\rho \in L^p(\mathbf{R}^+) \Rightarrow \mathcal{K}_1(\rho)(r) \leq ||\rho||_p C(q) r^q$. Especially $\mathcal{K}_1(\rho) \in L^q_{loc}$.

$\mathcal{K}_1$ ON LOCALLY BOUNDED ρ. $\rho(s) \leq M$ for $s \in [r/2, 2r]$ then $\mathcal{K}_1(\rho)(r) \leq 2Mr + 2c$, where $c = \rho(\mathbf{C})$.

POTENTIAL BOUND. $-(u_\rho(r) - C_\rho(r)) \leq \mathcal{K}(\rho)(r) \leq -2(u_\rho(r) - C_\rho(r))$ and $\mathcal{K}_1(\rho)(r) \leq (u_\rho(r) - C_\rho(r))$.

$*$-FUNCTION: $u_\rho(r) = \int_0^\infty \log|r-s|\; d\rho(s)$
$C_\rho(r) = (2\pi)^{-1}\int_0^{2\pi} u_\rho(r\exp(i\theta))\; d\theta$

PROOF L^p CASE: use Hölder inequality $\int_0^r (\frac{t}{r})^m \rho(t)dt \leq ||(\frac{t}{r})^m||_{L^q(0,r)}||\rho||_p = (\frac{r}{qm+1})^{1/q}||\rho||_p$ and $\int_r^\infty (\frac{r}{t})^m\rho(t)dt \leq ||(\frac{r}{t})^m||_{L^q(r,\infty)} = (\frac{r}{qm-1})^{1/q}||\rho||_p||\rho||_p$ Therefore $\int_0^r (\frac{t}{r})^m\rho(t)dt + \int_r^\infty (\frac{r}{t})^m\rho(t)dt \leq r^{1/q}(\frac{2qm}{q^2m^2-1})^{1/q}||\rho||_p \leq r^{1/q}(2/qm)^{1/q}||\rho||_p$. Sum up: $\mathcal{K}_1(\rho)(z) \leq ||\rho||_p|z|^{1/q}(2/q)^{1/q}\sum_{k=1}^\infty (2^{1/q})^{-k} = ||\rho||_p|z|^{1/q}(1/q)^{1/q}/(1-2^{-1/q})$.

EXAMPLES:
$p = \infty, q = 1$
$\mathcal{K}_1(\rho)(r) \leq ||\rho||_\infty r$.

$p = 2, q = 2$
$\mathcal{K}_1(\rho)(r) \leq (1+\sqrt{2})||\rho||_2\sqrt{r}$.

PROOF OF $*$-FUNCTION BOUND: left inequality by Homogenisation thm. Assume no atom at r:

$$
\begin{aligned}
\int_0^r \log|r-s|\; d\rho(s) &= \int_0^r \log|1-\frac{s}{r}|\; d\rho(s) + \int_0^r \log(r)\; d\rho(s) \\
\int_r^\infty \log|r-s|\; d\rho(s) &= \int_r^\infty \log|1-\frac{r}{s}|\; d\rho(s) + \int_r^\infty \log(s)\; d\rho(s)\;.
\end{aligned}
$$

$$
\begin{aligned}
\int_0^r \log|1-\frac{s}{r}|\; d\rho(s) &= -\sum_{k=1}^\infty \int_0^r (\frac{s}{r})^k/k\; d\rho(s) \\
\int_r^\infty \log|1-\frac{r}{s}|\; d\rho(s) &= -\sum_{k=1}^\infty \int_r^\infty (\frac{r}{s})^k/k\; d\rho(s)\;.
\end{aligned}
$$

$$\sum_{k=0}^\infty a^{2^k} \leq 2\sum_{k=1}^\infty \frac{a^k}{k}\;,\quad 0 < a < 1$$

because $a^2 \leq 2a^2/2, a^4 \leq 2(a^3/3 + a^4/4), a^8 \leq 2(a^5/5 + a^6/6 + a^7/7 + a^8/8)$, etc. Therefore, $\mathcal{K}(\rho)(r) = \sum_{k=0}^\infty \int_{\mathbf{R}^+} \exp(-|\log(r/t)|2^k)\; d\rho(t) \leq 2\sum_{k=1}^\infty \int_{\mathbf{R}^+} \exp(-|\log(r/t)|k)/k\; d\rho(t) = -2(\int_0^r \log|1-\frac{s}{r}|\; d\rho(s) + \int_r^\infty \log|1-\frac{r}{s}|\; d\rho(s)) = -2\int_0^\infty \log|r-s|\; d\rho(s) + 2\int_0^r \log(r)\; d\rho(s) + 2\int_r^\infty \log(s)\; d\rho(s))$ so that $\mathcal{K}(\rho)(r) \leq 2(-u_\rho(r) + \int_0^r \log(\frac{r}{s})\; d\rho(s) + u_\rho(0))$. Jensen's formula $\Rightarrow \mathcal{K}(\rho)(r) \leq 2(C_\rho(r) - u_\rho(r))$.

RADIAL MOLLIFIED SUBHARMONIC FUNCTION. $\phi(x) = \phi(1/x)$. Define $f_\phi(z) = \int_{\mathbf{R}} f(\beta z)\; d\phi(\beta)$ is potential if f is a potential: $dk_{f_\phi} = \int_0^\infty dk(\beta z)\; d\phi(\beta)$, $\rho_{f_\phi}(r) = \int_0^\infty \rho(\beta r) d\phi(\beta)$.

PROPERTIES. ρ_ϕ as smooth as ϕ and $|\rho_{f_\phi}(r)| \leq ||\phi_\infty||\rho(K_\phi)$. Worst case: ρ with atom of mass $\rho(K_\phi)$ at r.

MILLOUX-SCHMIDT. u subharmonic $\Rightarrow$ $\limsup_{r\to 0}\inf_\theta u(r\exp(i\theta)) = u(0)$. Proof: even for set of full Lebesgue density at 0 by Homogenisation theorem.

COROLLARY. $\exists r_n \to 0$ with $\limsup_{n\to\infty}\mathcal{K}(\rho)(r_n) = 0$ and $\forall\delta > 0$, $\{r, \mathcal{K}(\rho)(r) > -\delta\}$ has Lebesgue density 1 at 0.

APPLICATION (I). $g \in L^\infty(X, m)$ with $C = \int_{\mathbf{T}^1} \log|g(\theta)|\; d\theta < \infty$. T automorphism of $(X, \mathcal{A}, m)$. $E \in \mathbf{R}$. $A_\lambda(x) = \begin{pmatrix} E - \lambda g(x_1) & -1 \\ 1 & 0 \end{pmatrix}$ cocycle with Lyapunov exponent $\mu(A_\lambda)$.
Claim. $\exists\lambda_n \to \infty$ with $\limsup_{n\to\infty}\mu(A_{\lambda_n}) \geq \log|\lambda| + C$.

PROOF. $B_\epsilon(x) = \begin{pmatrix} \epsilon E + g(x) & -\epsilon \\ \epsilon & 0 \end{pmatrix}$. $\epsilon \mapsto \mu(B_\epsilon)$ is a subharmonic potential. $\mu(A_\lambda) = f(\epsilon) - \log|\epsilon| = f(\epsilon) + \log|\lambda|$. Have $C_f \geq 0$. Apply MILLOUX-SCHMIDT: $\exists\epsilon_n$ with $\limsup_n f(\epsilon_n) \geq 0$.

EXAMPLE: A transfer cocycle $A_{E=0}$ of $\tau + \tau^* + V'$ with C^2 Morse potential V has Lyapunov exponent $O(\log(\lambda))$ along subsequence $\lambda_n \to \infty$. Especially, $\exists$ no realanalytic V for which $\mu(A_{E,\lambda}) = O(1)$ for $\lambda \to \infty$.

APPLICATION (II) Setup as in (I). For $\delta > 0$, the set of parameters for which $\mu(A_\lambda) > \log|\lambda| + C - \delta$ has full Lebesgue density at $\lambda = \infty$.

PROOF. ϕ radial mollifier function. $\int_0^\infty \mu(A_{\lambda\beta})\; d\phi(\beta) \geq U := \log|\lambda| + \int_{\mathbf{T}} \log|g(x)|\; dx - 2|\phi|_\infty/\lambda$ and $\mu(A_\lambda) \leq V := c + \log(|2\lambda|) + C_1/|\lambda|$. x normalized Lebesgue measure of $\{\lambda \in [\lambda_0/2, 2\lambda_0],\ \mu(A_\lambda) = 0\}$. Have $(1-x) \geq U/V \Rightarrow x \leq C/\log(\lambda)$.