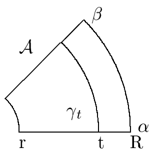
## 3. Talk: Subharmonicity, Jensen, Herman, Sorets-Spencer

ANALYTIC FUNCTIONS:
$U \subset \mathbf{C}$ open. $f \in C^1(U,\mathbf{C})$ is **analytic** in $U$ if $\partial f/\partial \overline{z} = 0$ in $U$. The equation $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}(f_x(x+iy) + if_y(x+iy)) = 0$ is called the **Cauchy-Riemann differential equation**.

EXAMPLES:
1) $f(z) = \sum_{j=1}^n a_j z^j$, analytic in $\mathbf{C}$ (entire).
2) $dk$ measure, compact upport $\mathrm{supp}(dk) \subset \mathbf{C} \Rightarrow f(z) = \int_{\mathbf{C}} \log(z-z')\, dk(z')$ analytic in complement of $\mathrm{supp}(dk)$.

JENSEN FORMULA:
$g$ analytic in $\mathcal{A} = \{te^{i\phi}, t \in [r,R],\ \phi \in [\alpha,\beta]\}$. Then, $\int_\alpha^\beta \log|g(Re^{i\phi})| d\phi = \int_\alpha^\beta \log|g(re^{i\phi})| d\phi + \int_r^R \frac{\mathrm{Arg}_{\gamma_t}(g)}{t} dt$, where $\mathrm{Arg}_{\gamma_t}(g) = \mathrm{Re}\frac{1}{2\pi i}\int_{\gamma_t} g'(z)/g(z)\ dz$ (argument change along $\gamma_t(\phi) = t\exp(i\phi)$, $\phi \in [\alpha,\beta]$).




PROOF: First: $g$ analytic, no roots. $\int_\alpha^\beta \log|g(Re^{2\pi ix})|\ dx = \int_\alpha^\beta \log|g(re^{2\pi ix})|\ dx + \int_{\mathcal{A}} \frac{\partial}{\partial t} \log|g(te^{2\pi ix})|\ dt\ dx$. Cauchy-Riemann equations $\frac{\partial}{\partial \overline{z}} g(z) = 0$. With $z = t\exp(2\pi ix), g'(z) = (\partial/\partial z) g(z)$, $\frac{\partial}{\partial t} \log|g(te^{2\pi ix})| = \mathrm{Re}\left(\frac{g'(te^{2\pi ix})}{g(te^{2\pi ix})} e^{2\pi ix}\right) = \mathrm{Re}\left(\frac{g'(z)}{g(z)}\ \frac{z}{t}\right)$. Using $2\pi i dx = dz/z$, $\int_\alpha^\beta \log|g(Re^{2\pi ix})|\ dx - \int_\alpha^\beta \log|g(re^{2\pi ix})|\ dx = \mathrm{Re}\frac{1}{2\pi i}\int_r^R \int_{\gamma_t} \frac{g'(z)}{g(z)}\ \frac{dz}{t}\ dt = \int_r^R \mathrm{Arg}_{\gamma_t}(g)\ \frac{dt}{t}$. In general, partition $\mathcal{A}$ into $n^2$ small sectors. Apply formula to sectors not intersecting $S$.

COROLLARY: JENSEN FORMULA
$g(z) = \prod_j (z-a_j) h(z)$, analytic $h$ with no roots $|z| \in [r,R]$:

$$\int_0^{2\pi} \log|g(Re^{i\theta})|\ d\theta = \int_0^{2\pi} \log|g(re^{i\theta})|\ d\theta = \sum_j \log(R/|a_j|)$$

PROOF. The Jensen formula is a quantitive version of subharmonicity.

$\mathrm{Arg}_{\gamma_t}(g)$ is an integer because $\gamma_t$ is closed which increments at $t = |a_j|$, which brings $\int_{|a_j|R} 1/t\ dt = \log(R/a_j)$.

EXAMPLE: FARADEY CAGE.
$L = a\tau + c\tau^* + b$, $a,b,c \in \mathbf{C}$ random operator on $l^2(\mathbf{Z})$ (constant coefficients). $F : l^2(\mathbf{Z}) \mapsto L^2(\mathbf{T})$ Fourier trafo diagonalizes $L$: because $FLF^*\hat{u}(\theta) = (a\exp(i\theta) + c\exp(-i\theta) + b)\hat{u}(\theta)$. For multiplication operators $\hat{L_f} : \hat{u} \mapsto \hat{f}u$ on $L^2(\mathbf{T})$ must have $\mathrm{tr}(L) = \int_{\mathbf{T}} f(\theta)\ d\theta$. For $|E| > ||L||$, where $\log\det(L-E) = \mathrm{tr}(\log|L-E|) = \log|E| + \mathrm{tr}(\log|1-L/E|) = \log|E| - \sum_{n=1}^\infty \mathrm{tr}((L/E)^n)/n$ have $\log\det(L-E) = \log\int_{\mathbf{T}}(f(\theta)-E)\ d\theta$ as only possible determinant. For $L = a\exp(i\theta) + c\exp(-i\theta) + b$, have $\log\det(L-E) = (2\pi)^{-1}\int_0^{2\pi} a\exp(i\theta) + c\exp(-i\theta) + b - E d\theta$. Jensen formula gives $\log\det(L-E) = 0$ for $E \in U$ elliptic region enclosed by spectrum of $L$ and else $\log\det(L-E) = \log|1-a_E^+|$, where $a_E^+$ is the root of $z \mapsto p(z) = az + cz^{-1} + b - E$ satisfying $|a_E^+| < 1$.

Lyapunov exponent of transfer cocycle $A_E = \begin{pmatrix} (E-b)/a & -c/a \\ 1 & 0 \end{pmatrix}$ of $L - E$ is $\log|\lambda_E^+|$, where $\lambda_E^+ = a_E^+$ is maximal eigenvalue of $A$ (with characteristic polynomial $p(z)/a$). DEFINE $\log\det(L-E) = \mu(A_E)$ for $E \in U$. 'Density of states' $dk = \Delta\mu(A_E)$ with support $\mathrm{supp}(dk) \subset U$.

HERMAN'S TRICK.
Let $w \mapsto A(w) \in SL(2,\mathbf{C})$ be analytic ($w \mapsto [A(w)]_{ij}$ analytic). $\mu(w) = \mu(A(w))$ Lyapunov exponent. Then $\int_0^1 \mu(\exp(2\pi i\theta))\ d\theta \geq \mu(0)$.

$n^{-1}\log||A^n(w,x)||$ subharmonic $\Rightarrow n^{-1}\int_X \log||A^n(w,x)||\ dm(x)$ subharmonic $\Rightarrow w \mapsto \mu(w) = \inf_n \int_X \log||A^n(w,x)||\ dm(x)$ subharmonic.

EXAMPLE: ALMOST MATHIEU.
$X = \mathbf{T}^1$, $T(x) = x + \alpha$, $\alpha$ irrational. $A_\lambda(x)$ transfer cocycle of almost Mathieu operator $L = \tau + \tau^* + \lambda\cos(x)$. Then, $\mu(A_\lambda) \geq \log(\lambda/2)$.

PROOF. $A_\lambda(x) = \begin{pmatrix} E - \lambda\cos(x) & -1 \\ 1 & 0 \end{pmatrix}$. $w = \exp(2\pi ix)$, $a = \exp(2\pi i\alpha)$, $\mathbf{T}^1 = \{|w| = 1\}$, $T(w) = az$ extends to $\mathbf{D} = \{|w| < r\}$, $r > 1$. Complex parameterized cocycle $A_\lambda(z,w) = \begin{pmatrix} E - \lambda(zw + (zw)^{-1}) & -1 \\ 1 & 0 \end{pmatrix}$. Analytic cocycle $B_\lambda(z,w) = \begin{pmatrix} Ewz - \lambda((wz)^2+1) & -wz \\ wz & 0 \end{pmatrix}$. Then $\mu(A_\lambda)(z) = \mu(B_\lambda)(z)$ for $|z| = 1$ and $\mu(B_\lambda)(z=0) = \log(\lambda/2)$. So, $\mu(A_\lambda) = \int_{|z|=1} \mu(A_\lambda(z)) \geq \log(\lambda/2)$.

EXAMPLE: STANDARD MAP $X = \mathbf{T}^2$, $T(x,y) = (2x - y + \lambda\cos(x), x)$. For $\alpha \in \mathbf{T}$ define $R_\alpha(x,y) = (x+\alpha, y+\alpha)$ and $T_\alpha = RTR^{-1}$. Jacobean cocycle $dT(x,y)$ of $T_0$ over $T_\alpha$. $(2\pi)^{-1}\int_0^{2\pi} \mu(dT, T_\alpha)\ d\alpha \geq \log(\lambda/2)$. Existes many maps for which cocycle $dT$ has positive Lyapunov exponent.

PROOF. $A_\lambda(x) = \begin{pmatrix} 2 + \lambda\cos(x) & -1 \\ 1 & 0 \end{pmatrix}$. $w_n = \exp(2\pi i x_n)$, $(x_n, y_n) = T^n(x,y)$. $z = \exp(2\pi i\alpha)$. Similar as in Mathieu, consider $B_\lambda(z, T^n(x,y)) = \begin{pmatrix} Ew_n z - \lambda((w_n z)^2 + 1) & -w_n z \\ w_n z & 0 \end{pmatrix}$.

EXAMPLE: TWIST MAP $X = \mathbf{T}^2$, $T(x,y) = (2x - y + \lambda V'(x), x)$, $V \in C^{2+\epsilon}(\mathbf{T})$ is Morse potential $V''(x) \neq 0$ for $V'(x) = 0$. Consider cocycle $A(z) = dT_{z\lambda}(x,y)$ over $T_\lambda$. Get $(2\pi)^{-1}\int_0^{2\pi} \mu(dT_{\lambda\exp(i\theta)})\ d\theta \geq \log(\lambda/2) + C$, where $C = \int_{\mathbf{T}} \log(|V'(\theta)|)\ d\theta$.

Morse condition so that $C > -\infty$.

EXAMPLE: HERMAN SPECTRUM. Let $A : X \mapsto SL(2,\mathbf{R})$, $A(x) = D(x) \circ R(\phi(x))$ with $D(x) = \begin{pmatrix} c(x) & b(x) \\ 0 & c^{-1}(x) \end{pmatrix}$, $c, c^{-1}, b \in L^\infty(X,m)$, $\phi : X \mapsto \mathbf{T}$ measurable. Define $R(z) = \begin{pmatrix} \frac{z+z^{-1}}{2} & \frac{z-z^{-1}}{2i} \\ -\frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} \in SL(2,\mathbf{C})$, $A(z) = AR(z)$. $B(z) = zA(z)$ is analytic. For $z = \exp(i\beta)$, have $A(\beta) = A(z) = AR(\beta)$, where $R(\beta) \in SO(2,\mathbf{R})$. $(2\pi)^{-1}\int_0^{2\pi} \mu(A(\beta))\ d\beta \geq F(b,c) := \int_X \log\sqrt{((c+c^{-1})^2 + b^2)/4}\ dm$.

$L = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$, $G = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$. $\mu(B_0) = \mu(DG) = \mu(L^{-1}DGL) = F(b,c)$. Use Herman trick.

SORETS-SPENCER (I). $X = \mathbf{T}^1$, $T(x,y) = x + \alpha$ irrational. $f$ realanalytic. $L_{\lambda\cos+\epsilon f}(x) = \tau + \tau^* + \lambda\cos(x) + \epsilon f(x)$, perturbation of almost Mathieu. $A_E = A_{E,\lambda\cos+\epsilon f}$ transfer cocycle of $L_{\lambda\cos+\epsilon f}$. $\mu(A_E) > \log(\lambda/2 - \delta)$ for small $\epsilon$ and $\forall E \in \mathbf{R}$.

PROOF. $\forall E, \lambda, \delta > 0, \exists r \in (0,1)\ \inf_{|z|=r} |\lambda/2 + z^2\lambda/2 - Ez| \geq \lambda/2 - \delta + r$. (For $r = |z| = 0$, inequality holds.) Define $g_n(z) = [A^n(z)]_{11}$ and $h_n(z) = [B^n(z)]_{11}$, with $B(z) = zA(z)$ which is analytic. Have $||A^n(z)|| \geq |g_n(z)| = |h_n(z)|$ for $|z| = 1$ so that $\mu(A) = \lim_{n\to\infty} n^{-1}\int_{|z|=1} \log|h_n(z)|\ dz$. With above $r, \delta \Rightarrow |h_n(z)| \geq (\lambda/2 - \delta)^n$ for all $n \in \mathbf{N}, |z| = r$. Proof: $P_j(z) = z_j E + (\lambda/2)(z_j^2 + 1)$, $z_j = T^j(z) = ze^{ij\alpha}z$. For $\eta \in \mathbf{C}$, define recursively $\eta_{j+1}(z) = (P_j(z_j)/z_j - \eta_j(z_j))^{-1}$. If $|\eta_j| < 1$, then $|\eta_{j+1}| < 1$. Have $h_n(z) = \prod_{j=1}^n (P_j(z) - z_j\eta_j)$ and $|P_j(z) - z_j\eta_j| \geq \lambda/2 - \delta$. Have $\mu(z) \geq \log(\lambda/2 - \delta)$ for $|z| = r$. Apply Jensen formula in $\{|z| \in [r,1]\}$ saying $\int_{|z|=1} \log|h_n(x)|\ dx \geq \int_{|z|=r} \log|h_n(x)|\ dx - \mathrm{Arg}_{|z|=r}(h_n)\log(r)$. Because $\mathrm{Arg}_{|z|=r}(h_n) = 0$ get $\mu(z) \geq \log(\lambda/2 - \delta)$ for $|z| = 1$.

SORETS-SPENCER (II). $X = \mathbf{T}^1$, $T(x,y) = x + \alpha$, irrational. $f$ realanalytic $f(\mathbf{R}) = [-1,1]$). $L_{\lambda f}(x) = \tau + \tau^* + \lambda f(x)$. Its transfer cocycle $A_E$ satisfies $\mu(A_{E,\lambda f}) > 0$ $\forall\lambda > \lambda_0, \forall E \in \mathbf{R}$.

PROOF. Need only consider $E \in I = [-||L||, ||L||]$. Realanalytic $f \Rightarrow \exists r > 0$ and $m > 0$ such that $|f - E/\lambda| \geq m > 0$ for all $|z| = r, E \in I, \lambda > \lambda_0$. From $|\lambda f - E| > m\lambda$ get $|h_n(z)| \geq \prod_{j=1}^n |P_j(z) - r| \geq r^n(m\lambda - 1)^n$, where $P_j(z) = z_j E + \lambda/2 f(z_j) z_j$. Also, $\mathrm{Arg}_{|z|=r} h_n(z) = \mathrm{Arg}_{|z|=r}\prod_{j=1}^n |P_j(z) - z_j\eta_j|$ $= \sum_{j=1}^n \mathrm{Arg}_{|z|=r} P_j(z) = n + n\mathrm{Arg}_{|z|=r}(\lambda f - E)$. Jensen formula gives $\mu(A_E) \geq \log(r(m\lambda - 1)) - \log(r)\mathrm{Arg}_{|z|=r}(\lambda f - E) = \log(m\lambda - 1) - \log(r)\mathrm{Arg}_{|z|=r}(\lambda f - E)$ which is $> 0$ for large $\lambda$ because $\mathrm{Arg}_{|z|=r}(\lambda f - E)$ is there $\lambda$ independent.

RIESZ MEASURE. $u = \log|z - a| \Rightarrow \Delta u = 2\pi\delta_a$ (Dirac measure at $a$). $u = \int_{\mathbf{C}} \log|z - w|\ dk(w) \Rightarrow \Delta u = 2\pi dk$.

$\int_{\mathbf{C}} \log|z|\Delta\phi(z) r dr d\theta = \lim_{r\to 0}\int_{|z|>r} \log(t)\Delta\phi(te^{i\theta}) t dt d\theta = \lim_{r\to 0}\int_0^{2\pi} \phi(re^{i\theta}) + r\log(r)\partial_r\phi(re^{i\theta})\ d\theta = 2\pi\phi(0)$. General case: integrate with respect to $dk$.

EXAMPLE. $\Delta\log^+|z| =$ Lebesgue measure on $\mathbf{T} = \{|z| = 1\}$.

EXAMPLE. $f(z)$ subharmonic, analytic for $|z| > R$ and $f(|z|) - a\log|z| \to 0$ for $|z| \to \infty$, then $f(z) = \int \log|z - z'|\ dk(z')$, with $dk(\mathbf{C}) = a, \mathrm{supp}(dK) \subset \{|z| < R\}$.

WEYL LEMMA. $u, v$ subharmonic in $U$, both not identically $-\infty$, $\Delta u = \Delta v \Rightarrow u = v + h$, with $h$ harmonic.

$U_r = \{z \in U \mid \mathrm{dist}(z, \delta U) > r\}$. $\chi_r$ $C^\infty$ smoothing function supported in $\{|z| < r\}$. Now $\Delta(u \star \chi_r) \in C^\infty(U_r)$ and $\Delta(u \star \chi_r) = \int \chi_r(z - w)\Delta u dA(w)$ implies with $\Delta u = \Delta v$ that $\Delta(u \star \chi_r) = \Delta(v \star \chi_r)$ so that $u \star \chi_r = v \star \chi_r + h_r$. Because $h - r \star \chi_s = h_r$ on $U_{r+s}$, have $h_r = h_{r+s}$. Now $h = \lim_{r\to 0} h_r$ is defined on $U$.

RIESZ DECOMPOSITION THEOREM. $f$ subharmonic, $dk = (2\pi)^{-1}\Delta f$ Riesz measure of $f$, then $f = \int \log|z - z'|\ dk(z') + h$, harmonic $h$.

Define $g(z) = \int \log|z - z'|\ dk(z')$. Then $\Delta f = \Delta g$ and by Weyl's lemma, $f = g + h$, where $h$ is harmonic.