2. Talk: Thouless formula (II), higher order difference operators

RANDOM JACOBI MATRIX:
is a random difference operator $L = a\tau + a(T^{-1})\tau^* + b$ with $a, b, a^{-1} \in L^\infty(X, \mathbf{C})$. No self-adjointness necessary.

TRANSFER MATRIX. $A_{E,n}(x) = \begin{pmatrix} (E-b_n)/a_{n-1} & -a_{n-1} \\ 1/a_{n-1} & 0 \end{pmatrix}$, $a_n = a(T^n x), c_n = c(T^n x), b_n = b(T^n x)$. $a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n = E u_n \Leftrightarrow A_{E,n} \begin{pmatrix} a_{n-1}u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} a_n u_{n+1} \\ u_n \end{pmatrix}$.

THOULESS FORMULA: L random Jacobi matrix. For $|E| > ||L||$

$\mu(A_E) = \log(\det(L-E)) - \int_X \log|a(x)|\, dm(x)$

PROOF. Take logarithms in G-MK-M formula, divide by n and integrate over X. LHS converges to Lyapunov exponent, RHS to $\log\det(L) - \int_X \log|a(x)|\, dm(x)$ by Avron-Simon and Birkhoff.

GREENE-MACKAY-MEISS FORMULA.
$\mathrm{Tr}(A_E^n) - 2 = \mathrm{Det}(L-E)/\prod_{i=1}^n a_i$. where $L = L_1$, with

$$L_\xi = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & a_n/\xi \\ a_1 & b_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & b_{n-1} & a_{n-1} \\ a_n\xi & 0 & \cdot & a_{n-1} & b_n \end{pmatrix}.$$

$R = (2 - \mathrm{Tr} * A_E^n))/4$ called Greene Residuum.

PROOF. Roots of $\xi \mapsto \mathrm{Det}(L_\xi - E)$ are eigenvalues λ, λ^{-1} of A_E^n and so, $\mathrm{Tr}(A^n) = \lambda + \lambda^{-1}$. Have $\mathrm{Det}(L_\xi - E) = C(\mathrm{Tr}(A^n) - \xi + \xi^{-1})$. Because $\mathrm{Det}(L(\xi) - E) = -\prod_{j=1}^n a_j \xi + \dots \Rightarrow C = -\prod_{j=1}^n a_j$. Therefore, $\mathrm{Det}(L_\xi - E) = -\prod_{j=1}^n a_j(\mathrm{Tr}(A^n) - (\xi + \xi^{-1}))$. Put $\xi = 1$.

General Jacobi matrices, where a is different from 1 in general appear in discrete time Hamiltonian systems.

DISCRETE TIME HAMILTONIANS:

$l(x,x') : \mathbf{T}^2 \to \mathbf{R}$	generating function
$\mathcal{L}(q) = \sum_{j=n_1}^{n_2} l(q_j, q_{j+1})$	action functional
$l_1(q_j, q_{j+1}) + l_2(q_{j-1}, q_j) = 0$	Euler equations
$dq_n = q_{n+1} - q_n$	derivative
$p_{n+1} = l_2(q_n, q_{n+1})$	momentum
$h(q_n, p_{n+1}) = p_{n+1}dq_n - l(q_n, q_{n+1})$	Legendre trafo
h	discrete Hamilton function
$dq_n = h_{p_{n+1}}, dp_n = -h_{q_n}$	Hamilton equations
$[L]_{ij} = \partial q_i \partial q_j \mathcal{L}(q_1, \dots, q_n)$	Hessian (Jacobi matrix)
$Lu = 0$	discrete Jacobi equations

CONTINUOUS TIME HAMILTONIANS:

$F(x,y) : \mathbf{R}^2 \to \mathbf{R}$	Legendre function
$\mathcal{L}(q) = \int_{t-1}^{t_2} F(q, \dot{q})dt$	action functional
$d/dt F_{\dot{x}} = F_x$	Lagrange equations
$\dot{x}$	velocity
$p = F_y$	momentum
$H(q,p) = p\dot{q} - F(q, \dot{q})$	Legendre trafo
H	Hamilton function
$\dot{q} = H_p, \dot{p} = -H_q$	Hamilton equations
$L = \partial_t F_{\dot{x}\dot{x}} \partial_t + \partial_t F_{x\dot{x}} - F_{x\dot{x}}\partial_t - F_{xx}$	Hessian
$L\phi = 0$	Jacobi equations

STANDARD TYPE MAPS:

$l(x,x') : (x-x')^2/2 - V(x)$	generating functions
$q_{j+1} + q_{j-1} - V'(x) = 0$	Euler equations
$p_{n+1} = (q_{n+1} - q_n)$	momentum
$h(q_n, p_{n+1}) = p_{n+1}^2/2 + V(q_n)$	Legendre trafo
$dq_n = p_{n+1}, dp_n = -V'(q_n)$	Hamilton equations
$(Lu)_n = u_{n+1} + u_{n-1} - V''(q_n)u_n$	Jacobi operator

KINETIC + POTENTIAL ENERGY:

$F(x,y) : my^2/2 - V(x)$	Legendre function
$m\ddot{x} = V'(x)$	Newton equations
$p = my$	momentum
$H(q,p) = mp^2/2 + V(q)$	Legendre trafo
$\dot{q} = p, \dot{p} = -V'(q)$	Hamilton equations
$L = \partial_t^2 + V''(x)$	Schrödinger operator

EXAMPLE: Birkhoff billiard in convex closed smooth curve $x \mapsto \Gamma(x)$ of length 1. $l_i = l(q_{i-1}, q_i) = -|\Gamma(q_{i-1}) - \Gamma(q_i)|$, $p(x,x') = l_1(x,x') = \cos(\phi)$, ϕ impact angle, Euler equations: $l_1(q_n, q_{n+1}) + l_2(q_{n-1}, q_n) = 0$ 'angle of inpact = angle of reflection'. $a_i = l_{12}(q_i, q_{i+1}) = \sin(\phi_i)\sin(\phi_{i+1})/l_i$, $b_i = h_{11}(q_i, q_{i+1}) + h_{22}(q_{i-1}, q_i) = \sin^2(\phi_i)(1/l_i + 1/l_{i+1}) - 2\sin(\phi_i)\kappa_i$, where κ_i = curvature at $\Gamma(q_i)$.

GENERALIZED THOULESS FORMULA. L, K are Jacobi matrices, A_{LK} transfer cocycle of LK satisfying $A_{LK}(u_n, u_{n+1}, u_{n+2}, u_{n+3}) = (u_{n+1}, u_{n+2}, u_{n+3}, u_{n+4})$, $\mu(A_{LK})$ = sum of two largest Lyapunov exponents of A_{LK}. Then $\mu(A_{LK}) = \log(\det LK)$.

PROOF. Solutions to $LKu = 0$ are obtained if $Ku = 0$ and (u_n, u_{n+1}) is in Oseledec space (OS) of $A_K(T^n x)$ or if $LKw = 0$ and (v_n, v_{n+1}), where $v = Kw$ is chosen in OS of $A_L(T^n x)$. Sove $v = Kw$ for $w \in \mathbf{C}^{\mathbf{Z}}$ by recursion so that $w \neq 0$ is in OS of A_{LK}:(from arbitrary solution $\tilde{w}$ define $w_n = \lim_{m\to\infty} A_{LK}^m w_{n-m}$). Write $\mu^\pm(u) = \lim_{n\to\pm\infty} n^{-1}\log|u_n|$ if u is in an OS. Because w is in OS of A_{LK} have $\mu^+(w) = -\mu^-(w)$. Also $\mu^+(v) = -\mu^-(v)$. Because $v = Kw$ implies $v_n \leq ||K||(|w_n| + |w_{n-1}| + |w_{n+1}|)$, have $\mu^+(v) \leq \mu^+(w)$ and $\mu^-(v) \leq \mu^-(w)$ implying $\mu^+(v) = \mu^+(w)$. Both $\mu^+(u) = \mu(A_K) \geq 0, \mu^+(v) = \mu^+(w) = \mu(A_L) \geq 0$ are Lyapunov exponents of A_{LK}. Therefore $\mu(A_{LK}) = \mu(A_K) + \mu(A_L)$.

DEFINITION OF DETERMINANT, DENSITY OF STATES. Known determinant for $L - E$ with $|E| > ||L||$ can uniquely analytically continued for E in unbounded connected component of resolvent set of L. Define yet undefined 'density of states' of L for random Jacobi matrices L_i as Riesz measure dk of Lyapunov exponent and define determinant by $\log\det(L-E) = \mu(A_{L-E})$. RHS defined $\forall E \in \mathbf{C}$. Define $\det(\prod_{j=1}^n L_j) = \prod_{j=1}^n \det(L_i)$.

THOULESS FORMULA. L_i random Jacobi matrices, then $\log\det(L)$ is arithmetic mean of n largest Lyapunov exponents of the transfer cocycle $A_L(T^k)(u_k, \cdots, u_{k+2n}) = (u_{k+n}, \cdots, u_{k+3n})$. Definition independent of representation $L = \prod_{j=1}^n L_j = \prod_{j=1}^n K_j$.

EXAMPLE. $L = L^{(1)}L^{(2)}$ with Jacobi matrices $(L^{(k)}(x)u)_n = u_{n+1} + u_{n-1} + V_n^{(k)}(x_n^{(k)})u_n$.

$$L(x) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 \\ 1 & c_{n-2} & b_{n-1} & a_{n-1} & 1 & 0 & 0 \\ 0 & 1 & c_{n-1} & b_n & a_n & 1 & 0 \\ 0 & 0 & 1 & c_n & b_{n+1} & a_{n+1} & 1 \\ 0 & 0 & 0 & 1 & c_{n+1} & b_{n+2} & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot \end{pmatrix}.$$

Equivalent to Jacobi matrix $\tilde{L} = \tilde{a}\tilde{\tau} + (\tilde{c}\tilde{\tau})^* + \tilde{b}$, $(\tilde{L}u)_{\tilde{n}} = \tilde{a}_{\tilde{n}}u_{\tilde{n}+1} + \tilde{c}_{\tilde{n}-2}u_{\tilde{n}-1} + \tilde{b}_{\tilde{n}}u_{\tilde{n}}$, with matrix-valued entries $\tilde{b}_{\tilde{n}} = \begin{pmatrix} b_{n-1} & a_{n-1} \\ c_{n-1} & b_n \end{pmatrix}$, $\tilde{a}_{\tilde{n}} = \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix}$, $\tilde{c}_{\tilde{n}} = \begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix}$, where $a_n = V_n^{(1)}(x^{(1)}) + V_{n+1}^{(2)}(x^{(2)})$, $b_n = V_n^{(1)}(x^{(1)})V_n^{(2)}(x^{(2)}) + 2$, $c_n = V_n^{(1)}(x^{(1)}) + V_{n-1}^{(2)}(x^{(2)})$.

TRANSFER COCYLE. transfer cocycle $A_E = \begin{pmatrix} (E - \tilde{b}_{\tilde{n}})\tilde{a}_{\tilde{n}-2}^{-1} & -\tilde{c}_{\tilde{n}-2} \\ \tilde{a}_{\tilde{n}-2}^{-1} & 0 \end{pmatrix} =$

$$\begin{pmatrix} E - b_{n-1} + a_{n-1}a_{n-2} & -a_{n-1} & -1 & -c_{n-2} \\ a_{n-2}(b_n - E) - c_{n-1} & E - b_n & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -a_{n-2} & 1 & 0 & 0 \end{pmatrix}$$

satisfies $A_E \begin{pmatrix} \tilde{a}_{n-2}u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_n u_{n+1} \\ u_n \end{pmatrix}$ if $(\tilde{L}(\mathbf{x})u)_n = Eu_n$ for $\mathbf{C}^2$ valued sequences $u \in (\mathbf{C}^2)^{\mathbf{Z}}$.

EXTERIOR PRODUCT.
H, d-dimensional Hilbert space, basis $e_1, \cdots, e_d$. Define vector space $\Lambda^p H = \{\sum c_i u_{i_1} \wedge \dots \wedge u_{i_p} \mid u_{i_j} \in H\}$ with identifications $u_1 \wedge \dots \wedge (u_j + u_j') \wedge \dots \wedge u_p = u_1 \wedge \dots \wedge u_j \wedge \dots \wedge u_p + u_1 \wedge \dots \wedge u_j' \wedge \dots \wedge u_p$, $c(u_1 \wedge \dots \wedge u_j \wedge \dots \wedge u_p) = u_1 \wedge \dots \wedge cu_j \wedge \dots \wedge u_p$ $u_{\pi(1)} \wedge \dots \wedge u_{\pi(p)} = \mathrm{sign}(\pi)u_1 \wedge \dots \wedge u_j \wedge \dots \wedge u_p$. It has scalar product $< u_1 \wedge \dots \wedge u_j \wedge \dots \wedge u_p, v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_p >= \det(< u_i, v_j >)$.

EXTERIOR POWER.
$p-$ fold exterior power of linear map $A : H \to H$ is $\Lambda^p A : u_1 \wedge \dots \wedge u_j \wedge \dots \wedge u_p \mapsto Au_1 \wedge \dots \wedge Au_j \wedge \dots \wedge Au_p$. Clearly $(\Lambda^p A)(\Lambda^p B) = \Lambda^p(AB)$ and $(\Lambda^p A)^* = \Lambda^p A^*$. Eigenvalues of $\Lambda^p A$ are $\{\prod_{k=1}^p \lambda_{i_k} \mid 1 \leq i_1 < \cdots < i_p \leq d\}$, if $\lambda_1, \dots, \lambda_d$. are eigenvalues of A. Proof: bring into Jordan normal form.

OSELEDEC THEOREM.
$(X, \mathcal{A}, T, m)$ dynamical system. Notation $|A| = (AA^*)^{1/2}$ for matrix A. $A : X \to GL(d, \mathbf{R})$ cocycle with $\log^+ ||A^{\pm 1}|| \in L^1(X)$. Then $\overline{A} = \lim_{n\to\infty} |A^n(x)|^{1/n}$, exists almost everywhere. Let $U_1(x), \dots, U_{s(x)}(x)$ are eigenspaces of the different eigenvalues $\exp(\lambda_1(x)) < \exp(\lambda_2(x)) < \dots < \exp(\lambda_{s(x)}(x))$ of $\overline{A}(x)$ called Oseledec spaces. Define $V_0(x) = \{0\}, V_r(x) = \oplus_{i=1}^r U_i(x)$. For $v \in V_r(x) \setminus V_{r-1}(x)$, one has $\lim_{n\to\infty} \frac{1}{n}\log|A^n(x)v| = \lambda_r(x)$.

The proof of Oseledec's theorem as given by Ragunathan, Ruelle, Ledrappier relies on Kingman's subadditive ergodic theorem.

KINGMAN'S SUBADDITIVE ERGODIC THEOREM.
$f_n : X \to \mathbf{R} \cup \{-\infty\}$ subadditive: $f_{m+n} \leq f_m + f_n(T^m)$. Assume $f_n^+ := \max(0, f_n) \in L^1(X)$. There exists T-invariant measurable function $f : X \to \mathbf{R} \cup \{-\infty\}$ such that $\frac{1}{n}f_n(x) \to f(x)$ for m-almost all $x \in X$. Furthermore $\frac{1}{n}\int_X f_n(x)\, dm \to \int_X f(x)\, dm(x) \in \mathbf{R} \cup \{-\infty\}$.