1. Talk: Random difference operators, Thouless formula, Determinants

BASIC IDEA FOR LOWER BOUND ON ENTROPY OF STANDARD MAP:
1) Entropy $= \log\det(L)$, for some random difference operator L (Pesin, Thouless)
2) Estimate $\log(\det(L\hat{L})) \geq a$, with a suitable operator $\hat{L}$. (Homogenisation or Herman)
3) Estimate $\log(\det(\hat{L})) \leq c$ (Hadamard, Jensen)
4) $\log(\det(L) = \log\det(L\hat{L}) - \log\det(\hat{L}) \geq a - c$ (Kadison-Fuglede).

RANDOM OPERATORS:
$(X, \mathcal{A}, m)$ probability space. H Hilbert space, $\mathcal{B}(H)$ algebra of bounded linear operators on H. A map $L : X \mapsto \mathcal{B}(H)$ is called a **random operator** if $x \mapsto <\phi, L\psi>$ is measurable for all $\phi, \psi \in H$.

RANDOM DIFFERENCE OPERATORS:
$H = l^2(\mathbf{Z})$. Let $T : X \mapsto X$ a measurable invertible transformation. $(\tau u)_n = u_{n+1}$ shift, $(au)_n = a(T^n x)u_n$ random diagonal operator given by $a(x) \in L^\infty(X, m)$. A random operator $L = \sum_{j=-k}^{k} a_j \tau^j, a_j \in L^\infty(X, m)$, $(L(x)u)_n = \sum_{j=-k}^{k} a_j(T^n x)u_{n+j}$ is called a **random difference operator**. It is **ergodic** if T is ergodic.

EXAMPLES:
1) $L = a$, random diagonal operator.
1) $L = \tau$, unitary shift operator.
3) $d = \tau - 1$ first order difference operator (analogue of differential operator ∂_x.
4) $L = \tau - 2 + \tau^* = -dd^*$ **discrete Laplacian** (analogue of $-\Delta = -\partial_x^2$).
5) $X = \mathbf{T}$, $T(x) = x + \alpha$, $x_n = T^n(x)$, $L(x) = \tau + \tau^* + \lambda\cos(x_n)$, **almost Mathieu operator**.
6) $X = \mathbf{T}^2$, $T(x, y) = (2x - y + \lambda V(x), x)$ Standard map $(x_n, y_n) = T^n(x, y)$. $L(x, y) = \tau + w + \tau^* + \lambda V'(x_n)$, **Spencer operator**.
7) $X = \mathbf{T}^2$, $T(x, y) = (2x, y/2), x < 1/2, T(x, y) = (2x - 1, (y + 1)/2), x > 1/2$ Baker map, $L(x, y) = \tau + \tau^* + \lambda V(x_n)$, **Anderson operator**.

ALGEBRAIC RULES:
Treat random difference operators as polynomials in $\tau, \tau^* = \tau^{-1}$ and bring the τ, τ^*'s to the right using the rule $\tau a = a(T)\tau$, $\tau^* a = a(T^{-1})\tau^*$, where $a(T)$ is a short hand for $x \mapsto a(Tx)$. The adjoint operator is used as usual: i.e. $(a\tau)^* = \tau^* a^* = a^*(T^{-1})\tau^*$.

EXAMPLES:
1) $L = \tau + \tau^* + a$, $K = \tau - \tau^* + a(T)$, $LK = (\tau + \tau^* + a)(\tau - \tau^* + a(T)) = \tau^2 - \tau^2 + (a + a(T^2))\tau + aa(T)$.
2) $L = a\tau + (a\tau)^* + b$, $B = a\tau - (a\tau)^*$, $\dot{L} = [B, L] = a(b(T) - b)\tau + (a(b(T) - b)\tau)^* + 2(a^2 - a(T^{-1})^2)$. random Toda flow.

TRACE
Random difference operators have a **trace** $\mathrm{tr}(L) = \int_X [L(x)]_{00}\ dm(x)$. It is normalized, positive and $\mathrm{tr}(KL) = \mathrm{tr}(\sum_i a_i\tau_i \sum_j b_j\tau_j) = \int_X \sum_{i+j=0} a_i b_j(T^i)dm = \int_X \sum_{i+j=0} a_i(T^{-i})b_j dm = \int_X \sum_{i=-j} a_i(T^j)b_j dm = \mathrm{tr}(LK)$.

DENSITY OF STATES
For selfadjoint L, the functional $f \mapsto \mathrm{tr}(f(L))$ on $C_0(\mathbf{C})$ defines by Riesz representation theorem a Borel probability measure dk in the complex plane by $\mathrm{tr}(f(L)) = \int f(E)\ dk(E)$. The measure dk is called the **density of states** of L.

PROPERTIES:
dk is normalized: $\mathrm{tr}(1) = 1$ and is a positive measure: $f \geq 0$, $L(x) = \int_{\mathbf{C}} \lambda\ dE_x(\lambda)$ (spectral theorem) so that $f(L(x)) = \int_{\mathbf{C}} f(\lambda)\ dE_x(\lambda)$ and $[f(L(x))]_{00} = \int_{\mathbf{C}} f(\lambda)\ [dE_x]_{00}(\lambda) \geq 0$.

For nonselfadjoint operators, define $f(L)$ for analytic f by holomorphic functional calculus.

DETERMINANT:
Let L be a random difference operator. Define for $|E| > ||L||$ a **determinant** $\det(L - E)$ by $\log\det(L - E) = \mathrm{tr}(\log(|L - E|))$. For s.a. L, we have $\log\det(L - E) = \int_{\mathbf{C}} \log|E - E'|\ dk(E')$, the **logarithmic potential** of dk.

AVRON-SIMON LEMMA:
L ergodic random difference operator then $\mathrm{tr}(f(L^{(\Lambda)}(x))) \to \mathrm{tr}(f(L))$ for all holomorphic f and m-almost all $x \in X$. In general: $\int_X \mathrm{tr}(f(L^{(\Lambda)}(x)))\ dm(x) \to \mathrm{tr}(f(L))$.

NOTATION:
For $\Lambda = [-N, -(N-1), \dots, N-1, N]$ define the $(2N+1) \times (2N+1)$ matrix L^Λ by $[L^\Lambda(x)]_{ij} = L(x)_{ij}$ for $i, j \in \Lambda$. Define $\mathrm{tr}(L^\Lambda) = |\Lambda|^{-1}\sum_{k=-N}^{N} L^\Lambda_{kk}$.

PROOF. Enough to take $f(E) = E^n$. Compare

$$\mathrm{tr}((L^n)^\Lambda) = \frac{1}{|\Lambda|}\sum_{k \in \Lambda}\sum_{j_1, \dots, j_{n-1} = -\infty}^{\infty} [L]_{kj_1} \dots [L]_{j_{n-1}k}$$

$$\mathrm{tr}((L^\Lambda)^n) = \frac{1}{|\Lambda|}\sum_{k \in \Lambda}\sum_{j_1, \dots, j_{n-1} \in \Lambda} [L]_{kj_1} \dots [L]_{j_{n-1}k}$$

$\mathrm{tr}((L^n)^\Lambda) \to \mathrm{tr}(L^n)$ (Birkhoff). Because $L_{ij} = 0$ for $|i - j| > \rho$

$$\mathrm{tr}((L^n)^\Lambda) - \mathrm{tr}((L^\Lambda)^n) = \frac{1}{|\Lambda|}\sum_{k \in \Lambda}\sum_{\exists j_l \notin \Lambda} [L]_{kj_1} \dots [L]_{j_{n-1}k}$$

which is $\leq (2\rho + 1)^{n-1} 2(\rho + n)||L||^{n-1}/|\Lambda| \to 0$, $|\Lambda| \to \mathbf{Z}$.

Dynamical systems working Seminar, Oliver Knill, UT, **September 15, 1999**

APPROXIMATION:
$\det(L - E) = \lim_{\Lambda \to \mathbf{Z}} \mathrm{Det}(L^{(\Lambda)} - E)^{1/|\Lambda|}$ where Det is the usual determinant.

PRODUCT FORMULA:
Given L, K for which Avron-Simon applies. Then $\det(LK) = \det(L)\det(K)$.

EXAMPLES.
1) $L = a$, $\det(a - E) = \exp(\int_X \log(a(x) - E)\ dm(x))$.
2) $L = \tau$, $\det(\tau - E) = \exp(\log^+|E|)$, where $\log^+(x) = \min(x, 0)$. Note that $\mathrm{Det}(\tau^\Lambda) = 0$ but $\det(\tau) = 1$ violating Avron-Simon. (Deaper Reason: Weierstrass not valid $\to$ Mergellyan...).

LYAPUNOV EXPONENTS:
$(X, \mathcal{A}, T, m)$ dynamical system. $A : X \to SL(2, \mathbf{C})$ measurable, bounded matrix-valued map. $\mu(A) = \lim_{n\to\infty}\frac{1}{n}\int_X \log||A^n(x)||\ dm(x)$ is called the **Lyapunov exponent** of A, where $A^n(x) = A(T^{n-1}x)\cdots A(x)$.

The limit exists because $||A^{n+m}(x)|| = ||A^n(T^m(x))A^m(x)|| \leq ||A^n(T^m x)||||A^m(x)||$ implies that $b_n = n^{-1}\int_X \log||A^n||\ dm(x)$ is subadditive: $b_{n+m} \leq b_n + b_m$. Because $||A^n|| \geq 1 \Rightarrow b_n \geq 0$. For a subadditive sequence b_n of real numbers which is bounded below, b_n/n has a limit.

EXAMPLE: TRANSFER COCYCLES:
$(L(x)u)_n = u_{n+1} + u_{n-1} + V_n(x)u_n = Eu_n$, $V_n(x) = V(T^n(x))$ is equivalent to $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_E(T^n x)\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$, where $A_E(x) = \begin{pmatrix} E - V(x) & -1 \\ 1 & 0 \end{pmatrix}$. The Lyapunov exponent measures the exponential growth of generalized eigenfunctions u satisfying $L(x)u = Eu$.

EXAMPLE: JACOBEAN OF STANDARD MAP:
A map $T(x, y) = (Ex - y + V(x), x)$ on $\mathbf{T}^2$ with smooth V and $E \in \mathbf{Z}$ has the Jacobean $A(x, y) = dT(x, y) = \begin{pmatrix} E - V'(x) & -1 \\ 1 & 0 \end{pmatrix}$. The Jacobean dT^n of T^n is $A^n(x, y)$. The Lyapunov exponent $\mu(dT)$ measures the average sensitive dependence on initial conditions of T. By **Pesin's formula** it is equal to the entropy of T.

THOULESS FORMULA: $L = \tau + \tau^* + V$ transfer cocycle A_E.

(i) $|E| > ||L||$, $\mu(A_E) = \log\det(L - E)$.

(ii) L s.a. with density of states dk, $\Rightarrow \mu(A_E) = \log\det(L - E) = \int_{\mathbf{C}} \log|E - E'|\ dk(E')\ \forall E \in \mathbf{C}$.

PROOF. (i) $|E| > ||L||$. $f(E') = \log(E - E')$ is analytic for $|E'| \leq ||L||$. $A_E^n = \begin{pmatrix} P_n(E) & Q_{n-1}(E) \\ P_{n-2}(E) & Q_{n-2}(E) \end{pmatrix}$, where $P_n = E^n + \dots, Q_n = E^n + \dots$ are polynomials of degree n. $(P_n, Q_n$ depend on $x)$. Let $\Lambda_{k,n} = \{k, \dots n\}$. Have $P_n(E) = 0 \Leftrightarrow E$ is an eigenvalue of $L^{(\Lambda_{1,n})}$ and $Q_n(E) = 0 \Leftrightarrow E$ is an eigenvalue of $L^{(\Lambda_{2,n+1})}$. From $P_n(E) = \prod_j(E - E_j^{\Lambda_{1,n}})$, $Q_n(E) = \prod_j(E - E_j^{\Lambda_{2,n+1}})$ obtain $n^{-1}\log|P_n(E)| = \mathrm{tr}(L^{\Lambda_{1,n}})$ and $n^{-1}\log|Q_n(E)| = \mathrm{tr}(L^{\Lambda_{2,n+1}})$. By Avron-Simon, $n^{-1}\int_X \log|P_n(E)|\ dm(x) \to \mathrm{tr}(\log|L - E|)$ and $n^{-1}\int_X \log|Q_n(E)|\ dm(x) \to \mathrm{tr}(\log|L - E|)$. Because $\log(|P_n(E)|) \leq \log||A^n(x)|| \leq \log(|P_n(E)|^2 + |Q_n(E)|^2 + |P_{n-1}(E)|^2 + |Q_{n-1}(E)|^2)$ have $n^{-1}\int_X \log(|P_n(E)(x)|)\ dm(x) \to \mu(A_E)$.
(ii) $f(E') = \log(E - E')$ is analytic for $E \in \mathbf{C} \setminus \mathbf{R}$. Same proof as in (i). Avron-Simon gives $\mu(E) = \int_{\mathbf{C}} \log|E - E'|\ dk(E')$. These subharmonic functions are analytic on $\mathbf{C} \setminus \mathbf{R}$ and agree there. Agreement on $\mathbf{C}$ by Craig-Simon argument (see below).

SUBHARMONICITY:
$U \subset \mathbf{C}$ open. $f : U \to \mathbf{R}$ is **subharmonic** if **upper semicontinuous** ($\{f(z) < \alpha\}$ open $\forall\alpha$) and **submean inequality** $f(z) \leq (2\pi)^{-1}\int_0^{2\pi} f(z + re^{i\phi})\ d\phi$ holds for all $z \in U$ and r satisfying $\{w, |z - w| \leq r\} \subset U$.

EXAMPLES:
1) h, k subharmonic $\Rightarrow \max(h, k)$ subharmonic.
2) $f_{n+1} < f_n$ subharmonic $\Rightarrow f = \inf_n f_n$ subharmonic
3) f subharmonic $\Rightarrow f^2$ subharmonic.
4) h, k subharmonic, $a, b \geq 0 \Rightarrow ak + bh$ subharmonic.
5) g analytic $\Rightarrow f = \log|g|$ subharmonic.
6) $A : \mathbf{C} \mapsto SL(2, \mathbf{C})$ analytic $\Rightarrow \log||A||$ subharmonic.

CRAIG-SIMON ARGUMENT:
f subharmonic $\Rightarrow (\pi r^2)^{-1}\int_{|z-w| \leq r} f(w)\ dw \to f(z)$ for $r \to 0$.

PROOF. Can assume $z = 0$. Define $M_f(r) = \sup_{0 \leq \phi \leq 2\pi} f(re^{2\pi i\phi})$, $C_f(r) = (2\pi)^{-1}\int_0^{2\pi} f(re^{2\pi i\phi})\ d\phi$ and $B_f(r) = (\pi r^2)^{-1}\int_0^{2\pi}\int_0^r u(se^{it})sdsdt$. Now, $M_f(r) \geq C_f(r) \geq B_f(r) = (2/r^2)\int_0^r C_u(s)sds \geq f(0)$ and $\limsup_{r\to 0} M_f(r) \leq f(0)$ by upper semicontinuity.

EXAMPLE: $L = \tau + \tau^*$, $F : l^2(\mathbf{Z}) \mapsto L^2(\mathbf{T})$ Fourier trafo, $\hat{L} = ULU^* : L^2(\mathbf{T}) \mapsto L^2(\mathbf{T})$, $u(\theta) \mapsto 2\cos(\theta)u(\theta)$. Lyapunov exponent of $A_E = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$ is $\mathrm{Re}(\mathrm{arccosh}(E/2))$. Compare $\log\det(L - E) = \log\det(\hat{L} - E) = \int \log|2\cos(\theta) - E|\ d\theta = \mu(A_E)$. $L = \tau + \tau^* - 2$ is Spencer operator to integrable twist map $(x, y) \mapsto (2x - y, x)$. $E = -2$ is at bottom of spectrum of $\tau + \tau^* \Rightarrow$ minimal invariant tori etc. $L = \tau + \tau^* - 4$ is Spencer operator to Anosov twist map $(x, y) \mapsto (4x - y, x)$. $E = -4$ is outside spectrum of $\tau + \tau^* \Rightarrow$ invertible Hessian, structural stability by implicit function theorem, entropy bounds etc. In general $0 \notin \sigma(L) \Leftrightarrow T$ Anosov.