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SINGULAR CONTINUOUS SPECTRUM AND QUANTITATIVE RATES OF WEAK MIXING

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Abstract. We prove that for a dense G_δ of shift-invariant measures on $A^{\mathbf{Z}^d}$, all d shifts have purely singular continuous spectrum and give a new proof that in the weak topology of measure preserving $\mathbf{Z}^d$ transformations, a dense G_δ is generated by transformations with purely singular continuous spectrum. We also give new examples of smooth unitary cocycles over an irrational rotation which have purely singular continuous spectrum. Quantitative weak mixing properties are related by results of Strichartz and Last to spectral properties of the unitary Koopman operators.

1. **Introduction.**

A branch of ergodic theory deals with the relation of the dynamical properties of the transformation T and the spectral properties of the Koopman operator $U_T : f \mapsto f \circ T$ (see [6, 23, 5]). Especially, one aims to link ergodic properties of T (preserving a probability measure m) with the asymptotic behavior of the Fourier coefficients $(g, U_T^n g) = \hat{\mu}_n$ of spectral measures $\mu = \mu_g$ corresponding to zero-average m-square integrable functions. For example: a) absolute continuity of these measures implies mixing of T and b) $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n |c_k|^2 = 0$ for any g implies that T has no group translation as a factor.

Power law decays of the Césaro averages of the Fourier transform and the spectral properties of μ have been investigated by Strichartz [36, 37]. These results influenced the research in quantum dynamics [8, 4, 12, 21]), where the decay rate of the survival probability $|(g, U^n g)|^2$ is a measure for transport under the the quantum dynamical evolution $U = e^{iH}$ governed by a Schrödinger operator H (see [21] for more details). We look at these results in the context of ergodic theory.

First (in section 2) we review the question what happens Baire typically (with respect to the weak topology) in the set $\mathcal{G}$ of all transformations in the context of recent developments in operator theory [34, 33, 28, 27, 13, 11, 21]: The proof given here that generically the spectrum is singular continuous needs only Rohlin's lemma, Halmos conjugacy lemma and a recent result of Simon and establishes both genericity of weak mixing and genericity of singular spectrum.

Since a generic dynamical system is weakly mixing but not strongly mixing, a finer quantitative subdivision in the class of weakly mixing transformations seems desirable. We introduce in section 3 the notion of (uniform) weak h-mixing which is the property that there exists a constant C such that for (all) unit vectors f and unit vectors g in the ortho-complement of the constant functions, $n^{-1} \sum_{k=1}^n |(f, U^n g)|^2 \leq Ch(n^{-1})$. These properties are invariants of the dynamical system. There is no

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apparent relation with other mixing properties like partial, light or mild mixing. We reformulate Strichartz result as well as consequence of a recent RAGE theorem of Last in ergodic theory. In section 4, we show that a generic shift invariant measure gives a dynamical system which has purely singular continuous spectrum. This result as well as its generalization to higher dimensions in section 5) should be seen in the context of statistical mechanics. Section 6 deals with cocycles over an irrational rotation having purely singular continuous spectrum.

In an appendix, a result of Strichartz about Fourier transforms as well as the converse due to Last are proven in the case of Fourier series. Since the discrete reformulations do not follow from the Fourier transform versions, a translation of the original proofs is included.

2. **Generic purely singular continuous spectrum revisited.** An invertible measure preserving transformation (automorphism) T of a probability space $(X, \mathcal{A}, m)$ defines the unitary Koopman operator $U_T : f \mapsto f(T)$ on $L^2(X, m)$. Because 1 is always an eigenvalue, it is convenient to restrict U_T to the orthogonal complement $H_0 = \{f \mid \int f \, dm = 0 \}$ of the constant functions and to consider the spectrum of this operator. Let $\mathcal{G}$ be the complete topological group of automorphisms of $(X, \mathcal{A}, m)$ with the weak topology: T_j converges to T weakly, if $m(T_j(A) \Delta T(A)) \to 0$ for all $A \in \mathcal{A}$; this topology is metrizable and completeness is defined with respect to an equivalent metric ([9]).

Theorem 2.1. *In the weak topology on $\mathcal{G}$, a dense G_δ-set of transformations has purely singular continuous spectrum.*

Proof. By Halmos conjugacy lemma ([9], p.77), there is a dense set of weakly mixing transformations in $\mathcal{G}$. Rohlin's lemma ([9]) assures that the set of periodic and so pure point transformations is dense in the uniform topology and so in the weak topology. If $T_n \to T$ in $\mathcal{G}$, then $U_{T_n} \to U$ in the strong operator topology and so in the strong resolvent topology. We apply now Simon's "Wonderland theorem" [34, 33]) in the following form: let $\mathcal{X}$ be a complete metric space of unitary operators, where the metric is not weaker than the strong resolvent topology. If there exists a dense set with continuous spectrum and a dense set with pure point spectrum, then there exists a dense G_δ of operators with purely singular continuous spectrum. □

Remark 2.2. Theorem 2.1 was proven by Katok-Stepin in ([18], see also [17] remark 4.1.1) and by Choksi-Nadkarni in [3]. The result is stronger than Halmos genericity result ([9, 1]), which only establishes generic absence of point spectrum.

Remark 2.3. Related to Theorem 2.1 is the result that a generic volume preserving homeomorphism of a manifold has purely singular continuous spectrum ([19] p. 201) which strengthens Oxtoby-Ulam's theorem about generic ergodicity.

Remark 2.4. Examples of transformations with purely singular continuous spectrum are classes of interval exchange transformations ([14] Theorem 7.1). For other generic spectral properties of such systems see [38].

Remark 2.5. The Riemann-Lebesgue lemma implies that transformations with purely absolutely continuous (ac) spectrum are strongly mixing. The fact that strongly mixing is a set of first category ([9]) implies that pure ac spectrum is meager in $\mathcal{G}$ but does not at all exclude the existence of ac spectrum.

Remark 2.6. Theorem 2.1 implies Rohlin's result [29] that a generic transformation has zero metric entropy because a transformation with positive metric entropy contains some countable Lebesgue spectrum.

Remark 2.7. With the product topology induced on commuting transformations, the subset $\mathcal{G}^d \subset \mathcal{G} \times \cdots \times \mathcal{G}$ of $\mathbf{Z}^d$- actions is a complete metric space. Since all d projections from $\mathcal{G}^d$ to $\mathcal{G}$ are continuous and surjective, the inverse image of a generic set is generic and for a dense G_δ-set of $\mathbf{Z}^d$-actions, all the transformations $T^n, n \neq 0$ have purely singular continuous spectrum on H_0. This generalizes a result of Natarajan ([17] 4.1.1). Because systems with positive metric entropy have also for $d > 1$ some Lebesgue spectrum ([16]), a generic $\mathbf{Z}^d$-action has zero entropy.

Remark 2.8. Theorem 2.1 implies that a generic selfadjoint operator of the form $L = U_T + U_T^*$ on $L^2(X)$ has purely singular continuous spectrum because $U = L/2 + i\sqrt{1 - L^2/4}$ has similar spectral type then L. More generally one can show that for a generic $\mathbf{Z}^d$ action generated by $T_1, \dots, T_d : X \to X$, the operator $L = \sum_{i=1}^d U_{T_i} + U_{T_i}^*$ has purely singular continuous spectrum.

3. **A scale of weakly mixing dynamical systems.** In this section, a quantitative definition of weak mixing is introduced. It is motivated by recent developments in research on transport properties of quantum mechanical systems ([8, 4, 12, 21]).

Definition 3.1. Given a function $h : \mathbf{R} \to [0, \infty)$ satisfying $\lim_{\epsilon \to 0} h(\epsilon) = 0$. Call T uniformly weakly h-mixing, if there exists a constant C such that for all unit vectors $f, g \in L^2(X)$ and all n

$$n^{-1} \sum_{k=1}^{n} | \int \overline{f} g(T^k) \; dm - \int f \; dm \cdot \int g \; dm|^2 \leq C \cdot h(n^{-1}) \; . \tag{1}$$

If relation (1) holds for at least one vector $f \neq 0$ and all unit vectors g, we say T is partially weakly h-mixing.

Remark 3.2. "Uniform weak h-mixing" or "partially weak h-mixing" are invariants of dynamical systems in the same way as ergodicity, weakly mixing, mixing, entropy etc. We will just see that any weakly mixing system has partially weakly h-mixing for some h. (This does not directly follow from the definition because the decay rate function h for the spectral measure μ_g might be dependent on the vector g). The supremum over all h, for which T has partially weak h-mixing is an invariant of the dynamical system and it would be useful to have naturally defined examples of dynamical systems distinguished by this invariant.

Remark 3.3. Obviously, if $h_1 < h_2$ and T is weakly h_1-mixing, then it is weakly h_2-mixing. Especially, if $h(t) \leq t^{-1}$, a weakly h-mixing transformation has purely absolutely continuous spectrum and is therefore mixing.

Remark 3.4. The relation of weakly mixing with continuous spectrum is a reformulation of Wiener's theorem $n^{-1} \sum_{k=1}^n |(f, U^k f)|^2 \to \sum_{x \in \mathbf{T}} \mu_f(\{x\})^2$ which holds for $f \in H_0$.

Definition 3.5. A measure μ on the circle is called uniformly h-continuous, if there exists a constant C such that for all intervals I on $\mathbf{T}$ the inequality $\mu(I) \leq Ch(|I|)$ holds, where $|I|$ is the Lebesgue measure of μ.

Theorem 3.6 (Strichartz, Fourier series version)**.** *μ is a uniformly h-continuous measure on the circle. There exists a constant C such that for all $F \in L^2(\mathbf{T})$ and all n*

$$n^{-1} \sum_{k=1}^{n} |(\hat{F\mu})_k|^2 \leq C \cdot ||F||_2^2 \cdot h(n^{-1}) \; .$$

For the proof in the case of Fourier transform in $\mathbf{R}^d$ and $h(t) = t^{-\alpha}$ see [36]. A simplified proof in the case $d = 1$, $h(t) = t^{-\alpha}, \alpha \in (0, 1]$ in [21] can with some minor modification be adapted to the discrete time situation and is included in the Appendix.

Theorem 3.7 (Last's converse of Strichartz theorem, Fourier series version)**.** *If there exists C, such that $\frac{1}{n}\sum_{k=1}^{n} |\hat{\mu}_k|^2 < C{\cdot}h(n^{-1})$ for all $n \geq 0$, then μ is uniformly $\sqrt{h}$-continuous.*

A translation into the ergodic setup is:

Corollary 3.8. *If there exists a spectral measure μ_f of U_T which is uniformly h-continuous, then T is partially weak h-mixing. If T is partially weakly h-mixing, then there exists a spectral measure μ_f of U_T, which is uniformly $\sqrt{h}$-continuous.*

Proof. For any g in the cyclic subspace $H_f = \overline{\{U_T^n f \mid n \in \mathbf{Z}\}}$ of f, the spectral measure μ_g is of the form $\mu_g = F_g \mu_f$ with $F_g \in L^2(\mathbf{R})$. By Theorem 3.6,

$$n^{-1} \sum_{k=1}^{n} |\int_X \overline{f} g(T^k) \, dm|^2 \leq C \cdot ||g||_2 \cdot h(n^{-1}) \tag{2}$$

holds for $g \in H_f$. For g orthogonal to H_f, one trivially has $\int_X \overline{f} g(T^k) \, dm(x) = 0$. □

Remark 3.9. Given $g = 1_A - m(A) \in H_0$. If μ_g is uniformly h-continuous, then

$$n^{-1} \sum_{k=1}^{n} (m(A \cap T^k(A)) - m(A)^2)^2 \leq C h(n^{-1}) \, .$$

On the other hand, if this holds, then μ_{1_A} is uniformly $\sqrt{h}$-continuous.

Remark 3.10. The function h depends in general on g. If U_T had a hypercyclic unit vector g (the orbit $U^n f$ is dense on the unit sphere), then there exists h such that T is uniformly weakly h-mixing.

There exists a direct relation between decay rates of $\int \overline{f} f(T^n) \, dm$ for $f \in H_0$ and potential theoretical properties of the spectral measure μ_f. For $\alpha \in (0, 1)$, the potential theoretical α-energy of a spectral measure μ on $\mathbf{T}$ is

$$I_\alpha(\mu) = \int_{\mathbf{T}} \int_{\mathbf{T}} |\sin(\frac{x-y}{2})|^{-\alpha} \, d\mu(x) \, d\mu(y) \, .$$

Proposition 3.11. *μ_f has finite α-energy if and only if $\sum_{k=1}^{\infty} k^{\alpha-1} |\int \overline{f} f(T^k) \, dm|^2 < \infty$.*

Proof. $I_\alpha(\mu)$ is finite if and only if $\sum_k k^{\alpha-1} |\hat{\mu}_k|^2$ is finite because the Fourier transform of the kernel $\phi_\alpha(t) = |\sin(t/2)|^{-\alpha}$ satisfies $(\hat{\phi}_\alpha)_n = (C + o(1)) n^{\alpha-1}$ for some constant C ([15]). □

Remark 3.12. If the α-energy of a measure μ is finite, then it is α-continuous in the sense that $\mu(S) = 0$ for all sets S which have zero α-Hausdorff measure [7]. Also, the Hausdorff dimension of μ (the minimum of all Hausdorff dimensions of Borel sets S satisfying $\mu(S) = 1$) is bounded below by α if μ has finite α-energy (follows from [7] Theorem 4.13). Especially, if a spectral measure μ_g has zero Hausdorff dimension, then $\sum_{k=1}^{\infty} k^{\alpha-1} |\int \overline{g(x)} g(T^k x) \, dm(x)|^2$ is unbounded for any $\alpha > 0$.

Remark 3.13. For $g = 1_A - m(A)$, Proposition 3.11 says that a spectral measure μ_{1_A} has finite α-energy if and only if $\sum_{k=1}^{\infty} k^{\alpha-1}(m(A \cap T^k(A)) - m(A)^2)^2 < \infty$. A numerical determination of potential theoretical properties (and so an estimate of the Hausdorff dimension) of the spectral measures is in principle possible because $(m(A \cap T^k(A)) - m(A)^2)^2$ is accessible by experiments.

Remark 3.14. The kernel $\phi_\alpha(t)$ can be replaced by other kernels $\phi_h(t) = h(\sin(t/2))$ like for example the logarithmic kernel $\phi_0(t) = -\log(|\sin(t/2)|)$ which is in an appropriate sense the limit of ϕ_α for $\alpha \to 0$ and Proposition 3.11 stays true for $\alpha = 0$. The corresponding energy is the logarithmic potential theoretical energy of μ.

An almost immediate consequence of Wiener's theorem is the classical RAGE theorem in quantum mechanics saying that $n^{-1}\sum_{k=1}^{n} (U^k f, AU^k f) \to 0$ for all compact A if and only if f has a continuous spectral measure μ_f for the unitary U. The unitary U defines an evolution of probability measures with density $|f(k)|^2$ on $\mathbf{Z}^d$ where $f \in l^2(\mathbf{Z}^d)$ is a unit vector. If $Q_i u(n) = nu(n)$ are the position operators on $l^2(\mathbf{Z}^d)$, let $|Q|^2$ be the operator $\sum_i Q_i^2$. It is a consequence of RAGE that $n^{-1}\sum_{k=1}^{n}(U^n f, |Q|^2 U^n f) \to \infty$ if μ_f has some continuous component. New quantitative RAGE theorems have appeared in the last years [8, 4, 12, 21] (an example is Theorem 3.15 below which leads to the divergence $n^{-1}\sum_{k=1}^{n}(U^n f, |Q|^2 U^n f) \geq Cn^{2\alpha/d}$ if the projection of f to the subspace $H_\alpha := \overline{\{f \mid I_\alpha(\mu_f) < \infty\}}$ has positive length.)

Theorem 3.15 (Last's RAGE theorem, discrete version)**.** *If f is uniformly h-continuous for the unitary operator U of a Hilbert space H, then there exists a constant C such that for all operators A in the $p-$ Schatten class $n^{-1}\sum_{k=1}^{n} |(U^k f, AU^k f)_H| \leq C^{1/p}||A||_p h(n^{-1})^{1/p}$.*

With the discrete version of Theorem 3.6, the proof of [21] translates to the discrete case when replacing corresponding averaging integrals by sums.

Corollary 3.16. *If $f \in H_0$ has a uniformly h-continuous spectral measure for U_T, there exists a constant C such that for all $k \in L^2(X \times X)$*

$$n^{-1}\sum_{k=1}^{n} |\int_X \int_X f(T^k x)k(x,y)f(T^k y)\, dm(x)dm(y)| \leq C \cdot ||k||_{L^2(X\times X)}\sqrt{h(n^{-1})}\ .$$

Proof. The proof follows from Theorem 3.15 using that the integral operator $Kf(x) = \int k(x,y)f(y)\ dm(y)$ has a Hilbert-Schmidt norm $||K||_2$ that satisfies $||K||_2 \leq ||k||_{L^2(X\times X)}$. □

4. **Shift invariant measures.** Given a topological shift $(X = A^{\mathbf{Z}}, T)$ over a finite alphabet A. Consider the space $M(X)$ of shift-invariant measures with the weak-* topology.

Theorem 4.1. *There exists a dense G_δ of T-invariant measures m for which the abstract dynamical system (X, T, m) has purely singular continuous spectrum.*

Proof. Because changing the measure changes the Hilbert space on which the Koopman operator acts, we will construct a continuous map F from the space of shift invariant measures to the unitary group of operators on $l^2(\mathbf{Z})$ such that $F(\mu)$ restricted to the orthogonal complement of the eigenspace of 1 of $F(\mu)$ is unitarily equivalent to $U_T : H_0 \to H_0$ of the system (X, T, μ).

There exists a homeomorphism ϕ which maps X onto a subset $\phi(X)$ of $[0,1] \subset \mathbf{R}$. Let K be the set of shift-invariant measures on X. Define the set of Borel probability measures $M = \{1/2(\nu + \phi^*(\mu)) \mid \mu \in K\}$ on $\mathbf{R}$, where ϕ^* is the push-forward homeomorphism and where ν is the Lebesgue measure on $[-1, 0]$. (The combination of ν and $\phi^*(\mu)$ is taken so that the functions x^n are always linearly independent in $L^2([-1, 1], \mu)$ also if μ is a pure point measure with finitely many atoms.) For every $\mu \in M$, let $f_n^{(\mu)}$ be a basis in $L^2([-1, 1], \mu)$ obtained by the Gramm-Schmidt orthogonalisation of the linearly independent polynomials $p_n : x \mapsto x^n$ with respect to the scalar product $(f, g)_\mu = \int \overline{f} g \; d\mu$. Define a map $\tilde{F}$ from M to the space U of unitary matrices on the Hilbert space $l^2(N)$ by

$$\mu \mapsto \tilde{F}(\mu)_{nm} = \int_{[-1,1]} \overline{f_n^{(\mu)}(x)} f_m^{(\mu)}(Sx) \; d\mu(x) \; ,$$

where $S = \phi T \phi^{-1}$ is the conjugation of the shift T on $\phi(X)$ and $S(x) = x$ for $x \in [-1, 0]$. Let $F = \pi \tilde{F} \pi$, where π is the projection on the closed space of all functions in H which are vanishing on $[-1, 0]$ or μ almost everywhere constant in $[0, 1]$. While $\tilde{F}(\mu)$ contains the eigenvalue 1 with infinite multiplicity, $F(\mu)$ is conjugated to the unitary Koopman operator of the dynamical system (X, S, μ) expressed in a μ-dependent basis. For each n, the map $\mu \in M \mapsto f_n^{(\mu)} \in C([-1, 1])$ is continuous, because of the continuity of $\mu \mapsto (f, g)_\mu$ for fixed f, g and because the Gramm-Schmidt orthogonalisation is a finite process. Also $\mu \mapsto \int h \; d\mu$ is continuous for fixed $h \in C(X)$. An $\epsilon/3$ argument assures that $\mu \mapsto F(\mu)_{nm}$ is continuous for all m, n. Therefore, F is continuous as a map from M to the set of unitary operators on $l^2(\mathbf{Z})$. The image $F(M)$ is a complete metric space of unitary operators. Because a dense set of invariant measures give periodic dynamical systems [25] and a dense G_δ of invariant measures are weakly mixing [32], a dense set in $F(M)$ has pure point spectrum and a dense set of operators with continuous spectrum on H_0. By Simon's theorem, there is a generic set of operators with purely singular continuous spectrum on H_0. Because F is continuous and surjective and because $F(M)$ contains a dense G_δ of operators with purely singular continuous spectrum, the claim follows. □

Remark 4.2. Theorem 4.1 is not a corollary of Theorem 2.1 because there is no surjective continuous map ρ from the space of T-invariant measures to a topological space $\mathcal{G}$ of automorphisms of a probability space (Ω, m), such that (X, μ, T) is isomorphic to $(\Omega, m, \rho(\mu))$ because the image of ρ would be a compact subset in $\mathcal{G}$ containing only transformations with entropy $\leq \log |A|$.

Remark 4.3. A corollary of Theorem 4.1 is Sigmund's result [31] which says that the set of shift-invariant measures with zero metric entropy is a dense G_δ. A corollary of Theorem 4.1 that a generic shift invariant measure is weakly mixing [35].

Remark 4.4. An other corollary of theorem theorem 4.1 is that if T is an invertible Axiom-A diffeomorphism restricted to a basic setΩ_i, then there is a dense G_δ of T-invariant measures onΩ_i which have purely singular continuous spectrum (compare [30, 32]).

5. **Invariant measures of multi-dimensional shifts.** A $\mathbf{Z}^d$ dynamical system is a $\mathbf{Z}^d$ action of automorphisms of a probability space $(T, \mathcal{A}, m)$. We consider now $\mathbf{Z}^d$ dynamical systems obtained by taking a shift-invariant measure μ on $K^{\mathbf{Z}^d}$, where K is a compact metric space.

Theorem 5.1. *For a generic* $\mathbf{Z}^d$*-shift invariant measure in* $K^{(\mathbf{Z}^d)}$*, all shifts different from the identity have purely singular continuous spectrum.*

Proof. Theorem (4.1) stays true if the finite alphabet A is replaced by a compact metric space Y because there is a dense set of measures in $Y^{\mathbf{Z}}$ which have support on some $A^{\mathbf{Z}}$, where A is a finite subset of Y. We can especially take $Y = K^{(\mathbf{Z}^{d-1})}$, where K is a compact space. Let $M(Y)$ be the set of T_1 invariant measures on $X = K^{(\mathbf{Z}^d)}$. We know that a dense G_δ in $M_1(Y)$ has purely singular continuous spectrum. Each of the compact sets $M_k(Y) = \{|\Lambda_k|^{-1} \sum_{n \in \Lambda_k} (T^n)^* \mu \mid \mu \in M_1(Y)\}$ with$\Lambda_k = [-k, k]^d$ consists of T_1-invariant measures and each contains a dense G_δ of measures which have purely singular continuous spectrum with respect to T_1 (any of the sets contains a dense set of periodic measures and a dense set of weakly mixing measures so the proof in 4.1 leads to generic singular continuity). The intersection $M_\infty(Y)$ of all these sets $M_k(Y)$ is exactly the set of all $\mathbf{Z}^d$ invariant measures. It follows that a dense G_δ in $M_\infty(Y)$ has singular continuous spectrum with respect to the transformation T_1 (because if K_i are some closed subsets of a Baire space containing each a dense G_δ subset $H_i \subset K_i$, then $\bigcap_i H_i$ is a dense G_δ in $K = \bigcap K_i$.) Having a dense G_δ of $\mathbf{Z}^d$-invariant measures with purely singular continuous spectrum with respect to T_1, we obtain also a dense G_δ of $\mathbf{Z}^d$ invariant measures with purely singular continuous spectrum with respect to any of the shifts $T^n \neq Id, n \in \mathbf{Z}^d$. □

Remark 5.2. It follows with [16] that a generic shift invariant measure has zero entropy.

Remark 5.3. Let A be a finite set. A cellular automaton is a continuous map $\phi : X = A^{\mathbf{Z}^d} \to A^{\mathbf{Z}^d}$ which commutes with all shifts. Given a shift invariant measure μ on $A^{\mathbf{Z}^d}$ for which all shifts have purely singular continuous spectrum. The push-forward $\phi^*\mu$ has the same property because $(X,\phi^*\mu)$ is a factor of the system (X, μ) ([10]). There exists therefore a residual set of shift invariant measures on $\phi^k(X)$, for which the shifts have purely singular continuous spectrum.

Remark 5.4. Related to Theorem 5.1 is the open problem in crystallography whether there exist "turbulent but not chaotic crystals". A "crystal" is a point $x \in \{0,1\}^{\mathbf{Z}^d}$ for which the closure X of the orbit of the shift $\mathbf{Z}^d$ action is uniquely ergodic (and so minimal). A crystal is "turbulent", if the subshift has no nontrivial discrete spectrum and it is called "chaotic", if all the shifts have some absolutely continuous spectrum. Theorem 5.1 suggests (but does not imply) that the set of measures for which the shifts have singular continuous spectrum is generic in the set of measures with uniquely ergodic support.

6. **Circle-valued cocycles.** Let T_α be an irrational rotation $x \mapsto x + \alpha$ mod 1 on the circle $X = \mathbf{T}^1$. Define the maps $a_{\beta,s} : X \to \mathbf{T} = \{|z| = 1\}$ by $a_{\beta,s}(x) = e^{2\pi i s 1_{[0,\beta)}} x$.

Theorem 6.1. *Given $s \neq 0$. For a generic pair (α,β), the operator $(U_{\beta,s,\alpha} f)(x) = a_{\beta,s}(x) f(x + \alpha)$ has purely singular continuous spectrum.*

Proof. If α has bounded partial quotients and $s \neq 0$ and $\beta \notin \alpha\mathbf{R}/\mathbf{Z}$, then $U_{\beta,s,\alpha}$ has purely continuous spectrum ([24] Theorem 2.4) using [2] Proposition 2.1). If α is rational, then $U_{\beta,s,\alpha}$ has pure point spectrum. $U_{\beta,s,\alpha}$ depends continuously on (α,β) in the strong operator topology. The claim follows from Simon's theorem. □

Remark 6.2. Riley [26] has proven that for all irrational α, the unitary $U_{\beta,1/2,\alpha}$ has purely singular continuous spectrum for almost all β.

Let V be a measurable map from $\mathbf{T}^1 = \mathbf{R}/\mathbf{Z}$ to $\mathbf{R}$. Consider a differential equation $\dot{u} = H_t u$ for a function $u \in L^2(\mathbf{T}^1)$, where H_t is the time-dependent Hamiltonian $H_t = i\left(\frac{d}{d\theta} + V(\theta)\delta(t + 2\pi n\alpha)\right)$. The time one map is a unitary circle-valued cocycle.

Corollary 6.3. *There are potentials $V = s \cdot 1_{[0,\beta)}$ for which the unitary Floquet operator of H_t has purely singular continuous spectrum.*

Proof. The time one map is a unitary circle-valued cocycle of Theorem 6.1. □

Given $m \in \mathbf{Z}\setminus\{0\}$ and $a_{\phi,m}(x) = e^{2\pi i\phi(x)+mx}$, where ϕ is in the topological space $C(\mathbf{T})$ of continuous periodic functions. Denote by $U_{\phi,m,\alpha}$ the associated unitary weighted composition operator on $L^2(X)$: $(U_{\phi,m,\alpha}f)(x) = a_{\phi,m}(x)f(x+\alpha)$.

Theorem 6.4. *For $m \neq 0$, there would exists a dense G_δ of points $(\phi,\alpha) \in C(\mathbf{T}) \times \mathbf{R}$ such that the operator $U_{\phi,m,\alpha}$ has purely singular continuous spectrum.*

Proof. The complete metric space $C(\mathbf{T})$ contains the dense set of functions ϕ which have a Fourier series $\hat{\phi}_n = o(1/n)$. For such ϕ and irrational α, the operator $U_{\phi,m\alpha}$ has no eigenvalues if $m \neq 0$ ([22] Corollary 2). If α is rational, then the operator U has pure point spectrum. Apply Simon's theorem. □

Remark 6.5. In Theorem 6.4, the space $C(\mathbf{T})$ can be replaced with any subset of $L^2(\mathbf{T})$ which is equipped with a topology in which functions with Fourier series of order $o(1/n)$ are dense. Especially, there exist smooth cocycles with purely singular continuous spectrum.

Appendix: Fourier series version of a theorem of Strichartz and its converse of Last. Let $D_n(t) = \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$ be the Dirichlet kernel. The relation $\sum_{k=-n}^{n} |\hat{\mu}_k|^2 = \int_{\mathbf{T}} D_n(y-x)\ d\mu(x)d\mu(y)$ is used in the proof of Wiener's theorem ([20]). Because the Féjer kernels $K_n(t)$ satisfy

$$K_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{n+1}{2}t)}{\sin(t/2)}\right)^2 = \sum_{k=-n}^{n}(1 - \frac{|k|}{n+1})e^{ikt} = D_n(t) - \sum_{k=-n}^{n}\frac{|k|}{n+1}e^{ikt} ,$$

one gets

$$\begin{aligned} 0 &\leq \frac{1}{n+1}\sum_{k=-n}^{n}|k||\mu_k|^2 = \int_{\mathbf{T}}\int_{\mathbf{T}}(D_n(y-x) - K_n(y-x))d\mu(x)d\mu(y) \\ &= \sum_{k=-n}^{n}|\hat{\mu}_k|^2 - \int_{\mathbf{T}}\int_{\mathbf{T}}K_n(y-x)d\mu(x)d\mu(y) . \end{aligned} \tag{3}$$

Both of the following proofs are adaptations of proofs given in [21] in the case of Fourier transforms.

Proof of Theorem 3.7:

Proof. Because $\hat{\mu}_n = \overline{\hat{\mu}_{-n}}$, we can also sum from $-n$ to n, changing only the constant C. If μ is not uniformly $\sqrt{h}$ continuous, there exists a sequence of intervals $|I_k| \to 0$ with $\mu(I_l) \geq l\sqrt{h(|I_l|)}$. A property of the Féjer kernel $K_n(t)$ is that for large enough n, there exists $\delta > 0$ such that $\frac{1}{n}K_n(t) \geq \delta > 0$ if $1 \leq n|t| \leq \pi/2$. Choose n_l, so that $1 \leq n_l \cdot |I_l| \leq \pi/2$. Using Estimate (3), one gets

$$\sum_{k=-n_l}^{n_l}\frac{|\hat{\mu}_k|^2}{n_l} \geq \int_{\mathbf{T}}\int_{\mathbf{T}}\frac{K_{n_l}(y-x)}{n_l}\ d\mu(x)d\mu(y) \geq \delta\mu(I_l)^2 \geq \delta l^2 h(|I_l|) \geq \mathrm{C} \cdot h(\frac{1}{n_l}) ,$$

which contradicts the existence of C such that $\frac{1}{n}\sum_{k=-n}^{n}|\hat{\mu}_k|^2 \leq Ch(n^{-1})$. □

Proof of Theorem 3.6: $\frac{1}{n}\sum_{k=-n}^{n-1}|\hat{F\mu}|_k^2 \leq e\int_0^1 \sum_{k=-n}^{n-1}\frac{e^{-\frac{(k+\theta)^2}{n^2}}}{n}\,d\theta\,|\hat{F\mu}|_k^2$

$$
\begin{aligned}
&= e\int_0^1 \sum_{k=-n}^{n-1}\frac{e^{-\frac{(k+\theta)^2}{n^2}}}{n}\int_{\mathbf{T}^2} e^{-i(y-x)k}F(x)\overline{F(y)}\,d\theta d\mu(x)d\mu(y)\\
&= e\int_{\mathbf{T}^2}\int_0^1 \sum_{k=-n}^{n-1}\frac{e^{-\frac{(k+\theta)^2}{n^2}-i(x-y)k}}{n}\,F(x)\overline{F(y)}d\theta d\mu(x)d\mu(y)\\
&= e\int_{\mathbf{T}^2}\int_0^1 e^{-(x-y)^2\frac{n^2}{4}+i(x-y)\theta}\sum_{k=-n}^{n-1}\frac{e^{-(\frac{k+\theta}{n}+i(x-y)\frac{n}{2})^2}}{n}d\theta F(x)\overline{F(y)}d\mu(x)d\mu(y)\\
&\leq e\int_{\mathbf{T}^2} e^{-(x-y)^2\frac{n^2}{4}}\Big|\int_0^1 \sum_{k=-n}^{n-1}\frac{e^{-(i\frac{k+\theta}{n}+(x-y)\frac{n}{2})^2}}{n}\,d\theta\Big|\;|F(x)||F(y)|d\mu(x)\;d\mu(y)\\
&\leq e\int_{-\infty}^{\infty}\frac{e^{-(\frac{t}{n}+i(x-y)\frac{n}{2})^2}}{n}\,dt\int_{\mathbf{T}^2} e^{-(x-y)^2\frac{n^2}{4}}|F(x)|\;|F(y)|\;d\mu(x)d\mu(y)\\
&= e\sqrt{\pi}\int_{\mathbf{T}^2}(|F(x)||F(y)|)\cdot(e^{-(x-y)^2\frac{n^2}{4}})\;d\mu(x)\;d\mu(y)\\
&\leq e\sqrt{\pi}||F||_2^2(\int_{\mathbf{T}^2} e^{-(x-y)^2\frac{n^2}{2}}\;d\mu(x)\;d\mu(y))^{1/2}\\
&= e\sqrt{\pi}||F||_2^2(\sum_{k=0}^{\infty}\int_{k/n\leq|x-y|\leq(k+1)/n} e^{-(x-y)^2\frac{n^2}{2}}\;d\mu(x)\;d\mu(y))^{1/2}\\
&\leq e\sqrt{\pi}||F||_2^2 C_1 h(n^{-1})(\sum_{k=0}^{\infty}e^{-k^2/2})^{1/2}\leq C\cdot||F||_2^2\cdot h(n^{-1})\;.
\end{aligned}
$$

Note added in proof: A previous unpublished version of this paper without section 3) circulated since March 20, 1995 (see the $mp\_arc@@fireant.ma.utexas.edu$ document number 95-193).

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