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A new approach to Abel's integral operator and its application to stellar winds

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Abstract. Inverse problems are prevalent in astrophysics and many methods were developed for the reconstruction of these ill–posed equations. We present a new method for inverting Abel's integral equation by explicit diagonalization of the related Volterra operator. This method is particularly suitable for determining the velocity laws of stellar winds. We test the method for different frequently used parameterizations of velocity laws of stellar winds.

Key words: analytical methods – inverse problems – stellar winds

1. Introduction

Astronomy is not an experimental science where an object can be investigated by turning or rotate it in a desired position. Rather, the astronomer is usually left with a limited set of more or less accurate data, representing integrated quantities from a macroscopic system. Although these systems are usually governed by simple physical laws, the observed quantity is a convolution over the whole system. For example the emission of a collisionally dominated nebula is locally determined by the electron temperature and the density, but these quantities, we will call it the source function[1], cannot be expected to be constant over the emitting region. Thus, the observed flux, we call it data function, corresponds to an integral over the whole emission region. The determination of the local electron temperature for each point in the system, the source function, from the observed integrated quantity, the data function, is what we call an inverse problem.

Because of its observational nature, such inverse problems are prevalent throughout astronomy. Astrophysical examples can be found in the excellent text book of Craig & Brown (1986), and similar problems are encountered in image processing techniques. We give particular attention to the reconstruction of the density profile of the stellar wind in a binary system. This problem much resembles the X–ray tomography in medical science, where the X–ray photons absorbed by the various organs are measured for cross–sectional slices through the human body.

Although this problem seems to be well known, we found a new method which allows an accurate reconstruction of the source function. Our method is particularly suited for the reconstruction of the velocity profile of stellar winds due to the physical behaviour of the wind driving mechanism. The empirical determination of velocity laws for stellar winds is essential for providing important constraints to wind theories.

2. The problem

The geometry of our problem is depicted in Fig. 1. A compact, point–like radiation source shines through the wind of a companion star in a binary system. As a function of the orbital phase, the line of sight from the compact star to the observer passes the extended atmosphere of the companion at different impact parameters, b. In reality, such a situation occurs e.g. in the symbiotic binary EG And (Vogel 1991), where the wind producing companion is an M2.4 III giant with a radius of $\approx 75R_\odot$, and the radiation source is a hot pre–white dwarf with a radius of $\approx 0.02R_\odot$ (Vogel et al. 1992). Compared with the separation of the two stars of $\approx 320R_\odot$, the radiation source is indeed point–like.

The amount of attenuated light from the radiation source is proportional to the column density of scatterers or absorbers in the wind along the line of sight. In EG And the attenuation is due to Rayleigh scattering by neutral hydrogen. The optical depth is thus given by $n_H\sigma_\lambda(H)$, where $\sigma_\lambda(H)$ is the Rayleigh scattering cross–section for atomic hydrogen (see e.g. Isliker et al. 1989), and n_H is the column densities of neutral hydrogen. The observation of different column densities, n_H, as a function of phase thus provides, as in medical science, a "tomographic" view of the stellar wind.

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[1] In this paper the term source function should not be confused with the source function used in radiative transfer problems.

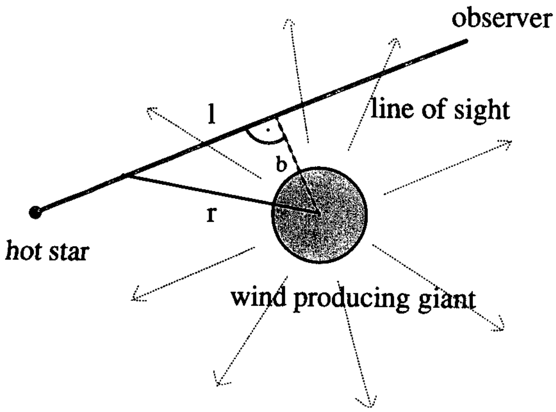

Fig. 1. The geometry of the problem is schematically presented

In practice, the observed column density is measured from the observed spectra as a function of the orbital phase. The impact parameter b, the projected separation between the two stars, is then given by

$$b^2 = p^2 \left(\cos^2 i + \sin^2\phi \sin^2 i\right) \quad , \tag{1}$$

where p denotes the separation between the two stars, ϕ is the orbital phase and i the inclination of the orbital plane. Our data function thus becomes

$$n_{\rm H}(b) = \int_{-\infty}^{\infty} N_{\rm H}(r)\, dl = \int_{-\infty}^{\infty} N_{\rm H}(\sqrt{l^2+b^2})\, dl \, , \tag{2}$$

where $N_{\rm H}(r)$ is the number density of atomic hydrogen at distance r from the mass losing star. Assuming a spherically symmetric neutral wind with a constant mass loss rate $\dot{M}$, the density at each point is determined by the mass loss rate of the star via the continuity equation :

$$\dot{M} = 4\pi r^2 \rho(r) v(r) = 4\pi r^2 N_{\rm H}(r) \mu m_{\rm H}\, v(r) \quad . \tag{3}$$

$\mu m_{\rm H}$ is the mean molecular weight in the wind, and $v(r)$ is the velocity of the stellar wind at distance r. The substitution of $N_{\rm H}$ from the continuity equation into Eq. (2) and a change of variable leads to :

$$n_{\rm H}(b) = a \int_b^{\infty} \frac{dr}{r v(r) \sqrt{r^2-b^2}} \, , \tag{4}$$

where

$$a = \frac{2\dot{M}}{4\pi \mu m_H} \, . \tag{5}$$

We use units such that the stellar radius and the terminal wind velocity are $R = 1$ and $v_\infty = 1$ respectively.

The problem consists of finding the run of the source function $v(r)$ from the observed data function $n_{\rm H}(b)$ by solving Eq. (4). This is an inverse problem which is notoriously ill–posed in the sense that their solutions are generally unstable against perturbations in the data function. Substituting $g(r) = a/(rv(r))$ in Eq.(4), we obtain the following form of Abel's integral equation

$$n(b) = A(g)(b) = \int_b^{\infty} \frac{1}{\sqrt{r^2-b^2}} g(r)\, dr \quad , \tag{6}$$

where A denotes the integral operator. The original form of Abel's integral equation is

$$\tilde{h}(v) = \tilde{A}(\tilde{g})(v) = \int_0^{v} \frac{1}{\sqrt{v-u}} \tilde{g}(u)\, du \tag{7}$$

which is like Eq. (6) a Volterra equation. By the change of variables $u = 1/r^2, v = 1/b^2$ and with the $\tilde{g}(u) = (r^2/2)\cdot g(r) = (1/2u)\cdot g(1/\sqrt{u})$, $\tilde{h}(b) = bn(b) = 1/\sqrt{v}\cdot n(1/\sqrt{v})$, one obtains Eq. (6).

The inversion of $\tilde{A}$ is given by

$$\tilde{g} = \tilde{A}^{-1}(\tilde{h}) = D\tilde{A}(\tilde{h}) \, , \tag{8}$$

where $D\tilde{h} = \tilde{h}'/\pi$ (see Cochran (1972) p.7 for a proof of Eq. (8)). Because $\tilde{A}^{-2} = D$, the inversion operator $\tilde{A}^{-1}$ is a square root of the differentiation and is like differentiation also numerically unstable. The explicit formula for $\tilde{A}^{-1}$ is not well suited for numerical calculations. There is also an analogue numerically bad inversion formula

$$A^{-1} = DAM \tag{9}$$

with $M(f)(y) = -2yf(y)$, (Minerbo & Levy 1969). For Eq. (4) this would lead to

$$\frac{1}{rv(r)} = \frac{1}{\pi a} \frac{d}{dr} A(-2b\cdot n_{\rm H})(r) \tag{10}$$

with a given in Eq. (5). In practice this inversion presents many problems due to the differentiation D. The Abel inversion problem is ill–posed because the inverse of the linear operator A is unbounded. One is forced therefore to bring the experimental data into a suitable linear space of functions, where the inversion is possible.

We will invert A here by diagonalising it explicitly, a procedure, which is not possible for the original integral operator $\tilde{A}$. This diagonalization is particularly advantageous for source functions that appear in velocity laws of stellar winds.

3. Diagonalization of Abel's operator

In order to diagonalize the linear operator A, we will show that the functions

$$\psi_i(r) = r^{-i}$$

are eigenfunctions of A for $i \geq 1$. In the case $i = 1$, this can be checked directly because

$$\int \frac{1}{r\cdot\sqrt{r^2-b^2}}\, dr = \frac{1}{b} \arccos\left|\frac{b}{r}\right| \tag{11}$$

and so

$$A(\psi_1)(b) = \int_b^\infty \frac{\psi_1(r)}{\sqrt{r^2-b^2}}\,dr = \frac{\pi}{2}\cdot\frac{1}{b} = \frac{\pi}{2}\psi_1(b)\,. \tag{12}$$

We come to the more complicated case $i \geq 2$. The change of variables $r \longmapsto u = \sqrt{r^2-b^2}$ and $dr \longmapsto du\frac{u}{\sqrt{b^2+u^2}}$ in Eq. (6) gives

$$n(b) = A(g)(b) = \int_0^\infty \frac{g(\sqrt{u^2+b^2})}{\sqrt{u^2+b^2}}\,du\,. \tag{13}$$

We now assume $g = \psi_i$ with $i \geq 2$. With this choice of i, the integral

$$\int_v^\infty A(\psi_i)(b)\,db = \int_v^\infty\int_0^\infty \frac{\psi_i(\sqrt{u^2+b^2})}{\sqrt{u^2+b^2}}\,du\,db \tag{14}$$

exists. Taking polar coordinates $\rho = \sqrt{u^2+b^2}$, $\phi = \arctan(\frac{u}{b})$, $du\,db = \rho\,d\rho\,d\phi$ gives

$$\int_v^\infty A(\psi_i)(b)\,db = \int_0^{\pi/2}\int_{v/\cos(\phi)}^\infty \psi_i(\rho)\,d\rho\,d\phi$$

$$= \int_0^{\pi/2} \frac{v^{-i+1}}{i-1}\cdot\cos^{i-1}(\phi)\,d\phi = \frac{v^{-i+1}}{i-1}\cdot\lambda_i$$

with

$$\lambda_i = \int_0^{\pi/2} \cos^{i-1}(\phi)\,d\phi\,. \tag{15}$$

We get after differentiation for $i \geq 2$

$$A(\psi_i) = \lambda_i\cdot\psi_i \tag{16}$$

This means that ψ_i is an eigenfunction of A with the eigenvalue λ_i.

Let now $g(r) = \sum_{i=1}^\infty g_i r^{-i}$ be an analytic function in $(0,\infty]$. In other words, we assume that for all $r > 0$, the Taylor series converge to a finite value $g(r)$ and $g(\infty) = 0$. The operator A maps g into a function n defined again on $(0,\infty]$ and having the Taylor expansion

$$n(b) = \sum_{i=1}^\infty n_i b^{-i}\,. \tag{17}$$

The coefficients of these Taylor expansions are related by

$$n_i = g_i\cdot\lambda_i\,. \tag{18}$$

The inversion of A is explicitly given by

$$A^{-1}:\quad \sum_{i=1}^\infty n_i b^{-i} \longmapsto \sum_{i=1}^\infty \frac{n_i}{\lambda_i} r^{-i}\quad. \tag{19}$$

We now have to find a suitable procedure for calculating the eigenvalues. Applying the inversion formula Eq. (9) to the eigenfunctions ψ_i gives for $i \geq 2$

$$\frac{1}{\lambda_i}\psi_i = A^{-1}\psi_i = DAM\psi_i = -2DA\psi_{i-1}$$

$$= -2\lambda_{i-1}D\psi_{i-1} = \frac{2}{\pi}(i-1)\lambda_{i-1}\psi_i$$

and this leads to the recursion formula

$$\lambda_i\lambda_{i-1} = \frac{\pi}{2i-2} \tag{20}$$

for the eigenvalues. From Eq. (12) we have $\lambda_1 = \pi/2$, thus, with the previous recursion one gets an explicit formula for $i \geq 2$ for the eigenvalues of A:

$$\lambda_i = \begin{cases} \frac{(i-2)!!}{(i-1)!!} & i \text{ even} \\ \frac{\pi}{2}\cdot\frac{(i-2)!!}{(i-1)!!} & i \text{ odd} \end{cases} \tag{21}$$

where $i!!$ denotes the double factorial of i which are recursively defined by $0!! = 1!! = 1, i!! = i\cdot(i-2)!!$. We note that these eigenvalues represent a monotonically decreasing series.

Our diagonalization again shows, that the Abel's operator is ill–posed, since the inverse of A is an unbounded operator. In other words, with the data function $n(b) = \sum_{i=1}^\infty n_i b^{-i}$ and a perturbation δn, the error in the source function g is

$$\delta g(r) = \sum_{i=1}^\infty \frac{\delta n_i}{\lambda_i} r^{-i}\,. \tag{22}$$

Since $\lambda_i \to 0$ as $i \to \infty$, δg can be arbitrary large even for infinitesimal data perturbations δn. An obvious way to make the problem stable is to truncate the sum in Eq. (22) before the eigenvalues become too small. We will use the properties of the source functions for winds, in order to obtain at the same time also a satisfactory accurate inversion method.

It is astonishing that the above explicit diagonalization of the operator A has not yet appeared in literature. We have checked textbooks about integral equations (Courant & Hilbert 1953; Cochran 1972; Sneddon 1972; Delves & Walsh 1974; Baker 1977; Fenyö & Stolle 1984; Kress 1989), general literature about inverse problems (Glasko 1984; Craig & Brown 1986), and especially about the Abel integral equation (Anderssen & De Hoog 1990), articles about new numerical methods for the Abel transform (Minerbo & Levy 1969; Kalal & Nugent 1988; Smith & Keefer 1988; Shimizu & Horigome 1989; Dong & Kearney 1991; Pretzler 1991), and dedicated papers dealing with the Abel integral equation in X–ray radiography (Deutsch et al. 1990), plasma spectroscopy (Sato 1987), electron–cyclotrons (Fidone et al. 1991), combustion diagnostics (Ladouceur & Adiga 1987), and tokamaks (Park 1989), but we found no sign for an approach similar to our explicit diagonalization. The interested reader is referred to the particularly extensive work about the Abel Integral equation of Anderssen & De Hoog (1990) and Craig & Brown (1986).)

An explanation for the absence of such a direct approach could be, that the operator A is usually reduced to the original form of Abel's operator, $\tilde{A}$, which allows no such diagonalization. An other reason could be the existence of explicit inversion formula Eq. (8) (already given by Abel 1826) or Eq. (9), which solve the inversion problem from the mathematical point of view. Nevertheless, we think that especially the recent need of good numerical inversion methods will make our explicit diagonalization in great esteem also with other applications.

4. The inversion of the stellar wind problem

We present in this section an inversion method that uses the eigenfunction decomposition found in the last section. But first we show that due to some natural properties of the source function $a/(rv(r))$, our method is well suited for the reconstruction of the velocity law of stellar winds.

4.1. Why it works

Wind velocities are commonly parameterized by a β–law:

$$v(r) = v_\infty \left(1 - \frac{R}{r}\right)^\beta . \tag{23}$$

This simple function with $\beta = 1/2$ was already used by Chandrasekhar (1934) for line profile calculations, but its victorious advance came with the pioneering paper by Castor et al. (1975) on the theory of radiation driven winds. They predicted a value of $\beta = 0.5$. The spectral analysis of hot stars yields empirical values of $0.5 \lesssim \beta \lesssim 1$ (see e.g. Lamers & Morton 1976; Hamann 1980; Schmutz et al. 1991). The β–law not only successfully describes the winds from hot stars. Schröder (1985) was also able to use it for the cooler K supergiants in ζ Aur / VV Cephei type binaries, where he derived shallower velocity laws with $2.5 \lesssim \beta \lesssim 3.5$. For the coolest mass losing stars, the M giants, there is only one empirical velocity determination so far (Vogel 1991), which requires an even shallower velocity law for the inner part of the wind, followed by an accelerated part at a few stellar radii. Such a velocity law may no longer be approximated by the β–law.

To test our inversion method for a range as wide as reasonable for potential stellar velocity laws, we investigate the cases, (*i*) $\beta = 0.5$ (adequate for hot stars), (*ii*) $\beta = 2.5$ (intermediate stars), and (*iii*) the velocity law given by Vogel (1991) for the M2.4 III giant in the symbiotic system EG And. These three velocity laws asymptotically reach the constant, terminal velocity for large distances. This qualitative behaviour is assumed for all stellar winds. This means, that the source function in our inversion problem asymptotically approaches the first eigenfunction a/r. Thus, for large distances the source function is well approximated by only the first eigenfunction, and the inversion is easily done. The same situation would be encountered for $\beta = 0$, where the whole inversion problem would shrink to a trivial equation.

Next, we consider a finite Taylor expansion of our source function:

$$\frac{1}{rv(r)} = g_1 \frac{1}{r} + g_2 \frac{1}{r^2} + g_3 \frac{1}{r^3} + \ldots + g_N \frac{1}{r^N} \tag{24}$$

The velocity is thus represented by

$$v(r) = \frac{r^{N-1}}{g_N + g_{N-1} r + \ldots + g_1 r^{N-1}} , \tag{25}$$

$g_1, \ldots, g_N$ are constants. Thus, for $r \to 0$ the velocity automatically goes to 0. The question is then, if the source function can also be well represented in an interval $[r_{min}, r_{max}]$ by a low order polynomial in r^{-i}. Physically, r_{max} must be in the region where $v(r)$ has reached its terminal velocity, and $r_{min} > R$ is the deepest point in the extended atmosphere, from which information can reach the observer. For all three considered velocity laws we have numerically confirmed, that the source functions are well represented in the interval $[r_{min}, r_{max}] = [1.2, 9.2]$ by a polynomial in r^{-1} of small order $L \le 8$.

4.2. The inversion method

We now turn to the actual inversion. From the exact data function n, we only know an experimentally determined data set

$$\{(b_i, f_i, \sigma_i) \mid i = 1, \ldots N,\ 0 < b_i \le \infty\} , \tag{26}$$

where σ_i denotes the individual measurement error. Note that in our physical problem $f_i = \infty$ for $b < 1$. We first fit this "experimental" data set by the polynomial

$$n^{(K)}(b) = \sum_{i=1}^{K} n_i^{(K)} b^{-i} \tag{27}$$

of order K that minimizes the merit function

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{n^{(K)}(b_i) - f_i}{\sigma_i} \right)^2 . \tag{28}$$

If $K \ge N$ then $\chi^2 = 0$.

To solve the inversion problem we only have to transform the polynomial $n^{(K)}$ by A^{-1} (see Eq. (19)) into

$$g^{(K)}(r) = \sum_{i=1}^{K} g_i^{(K)} r^{-i} = \sum_{i=1}^{K} \frac{n_i^{(K)}}{\lambda_i} r^{-i} , \tag{29}$$

which is our approximation to the real source function g.

5. Error analysis

It is well known that a least square procedure may lead to unphysical, highly oscillatory solutions. The unsatisfactory results obtained using such classical techniques lead to the development of smoothing methods, which use prior knowledge on the form of the searched source function (see Craig & Brown 1986; Anderssen & De Hoog 1990; Thompson & Craig 1992, and references therein). We overcome this problem by truncating the inverse operator, which is equivalent to the assumption that the source function is well approximated by a low order polynomial. In this chapter we estimate the error in the reconstructed source function obtained by this approximation. We consider two different possible errors in the data function or in its measurement by distinguishing an absolute and a relative error. In the fitting procedure discussed in the previous section, this will be reflected in the choice of the σ_i's.

5.1. Data functions with absolute errors

The error in the reconstructed source function depends on the error in the data function and on the inversion method. We first consider data containing a random absolute error. The merit function (28) gives an absolute error if we set $\sigma_i = 1$ for all i. Such a situation is encountered for example in the problem treated in the classical paper of Minerbo & Levy (1969). We thus assume random equally distributed errors obeying

$$|f_i - n(b_i)| \leq \epsilon_a \qquad \text{for all} \quad i \quad . \tag{30}$$

We designate by δ_a the bound on the absolute error

$$\left|f_i - n^{(K)}(b_i)\right| \leq \delta_a \qquad \text{for all} \quad i \quad , \tag{31}$$

which is a complicated function of the fitting polynomial $n^{(K)}$, the chosen impact parameters $b_1, b_2, \ldots, b_N$, the value of ϵ_a and the statistical distribution of the random error. If $K \geq N$ then $\delta_a = 0$.

The error of the reconstructed source function is measured by the standard deviation

$$\sigma_a = \left(\frac{E_a}{N}\right)^{1/2} , \tag{32}$$

where E_a is the sum:

$$E_a = \sum_{i=1}^{N} \left(g^{(K)}(b_i) - g(b_i)\right)^2 , \tag{33}$$

evaluated at the impact parameters of the observational data set. As defined by Eq. (29), $g^{(K)}$ is our reconstructed approximation for the true, unknown source function $g(r)$. If the true source function $g(r)$ is a polynomial of order L in r^{-1}, then the true data function $n(b)$ is via Eq. (16) also a polynomial of order L in b^{-1}. Writing the true data function $n(b)$ as the sum

$$n(b) = \sum_{i=1}^{L} n_i b^{-i} , \tag{34}$$

the true source function is given by

$$g(r) = \sum_{i=1}^{L} \frac{n_i}{\lambda_i} r^{-i} . \tag{35}$$

We can now substitute Eq. (29) and Eq. (35) into Eq. (33). By M we designate the maximum of L and K, this requires $n_{L+1} = \ldots = n_M = 0$ in the case $L < M$ and $n^{(K)}_{K+1} = \ldots = n^{(K)}_M = 0$ in the case $K < M$. We recall, that the eigenvalues are monotonically decreasing. The error of the reconstructed source function is thus

$$\begin{aligned} E_a &= \sum_{j=1}^{N} \left(\sum_{i=1}^{M} \frac{n_i^{(K)} - n_i}{\lambda_i} b_j^{-i}\right)^2 \\ &\leq \frac{1}{\lambda_M^2} \sum_{j=1}^{N} \left(n^{(K)}(b_j) - n(b_j)\right)^2 \\ &\leq \frac{1}{\lambda_M^2} \left(\sum_{j=1}^{N} (n^{(K)}(b_j) - f_j)^2 + \sum_{j=1}^{N} (n(b_j) - f_j)^2\right) \\ &+ \frac{1}{\lambda_M^2} \left(2 \sum_{j=1}^{N} (n(b_j) - f_j)(n^{(K)}(b_j) - f_j)\right) \\ &\leq \frac{1}{\lambda_M^2} \left(\chi^2 + N\epsilon_a^2 + 2N\epsilon_a\delta_a\right) , \end{aligned}$$

where δ_a and ϵ_a are given in Eqs. (31) and (30). From the last equation we obtain a bound on the standard deviation σ_a

$$\sigma_a \leq \frac{1}{\lambda_M} \left(\frac{\chi^2}{N}\right)^{1/2} + \frac{\epsilon_a}{\lambda_M} + \frac{\sqrt{2\epsilon_a\delta_a}}{\lambda_M} \quad . \tag{36}$$

We see that the bound on the standard deviation σ_a decreases as K increases, when $K \leq L$, because δ_a and χ^2 are decreasing functions of K (see Eqs. (28), (31)). The noise amplification term, the second term in the RHS of Eq. (36) is independent of K for $K \leq L$ as $M = \max(L, K) = L$. When $K \geq L$, then $M = K$ and χ^2 and δ_a continue to decrease, but the noise amplification term will increase because $\lambda_K \to 0$.

5.2. Data functions with relative errors

In our case of Rayleigh scattering in the wind of a companion star the attenuation of the light is proportional to $e^{-n(b)}$, and the observed column density may range over several magnitudes (see Fig. 2 of Vogel 1991). Therefore, the error in the observations is in this case rather a relative one:

$$\left|\frac{f_i - n(b_i)}{n(b_i)}\right| \leq \epsilon_r \qquad \text{for all} \quad i \quad . \tag{37}$$

Thus, for the fitting procedure we choose $\sigma_i = \epsilon_r \cdot n(b_i)$. In this situation the corresponding error in the reconstructed source function can only be determined numerically. This will be done in the next section.

6. Numerical tests

As already discussed in Sect. 4.1 we have chosen three potential velocity laws for stellar winds for testing our inversion method.

- The β law with $\beta = 0.5$ representing hot stars.
- The β law with $\beta = 2.5$ derived for K supergiants.
- The empirically determined velocity law for the cool M2.4 III giant in EG And (Vogel 1991).

For all three velocity laws the corresponding data function ranges over several magnitudes. It is therefore only meaningful to consider the case where the data function is influenced by a relative error. In the following we discuss the numerical simulation of the inversion and the error estimate.

For each of the 3 velocity laws mentioned above we repeated the same procedure:

(i) We numerically integrate the assumed velocity law $v(r)$ for $N = 40$ impact parameters b_i in the interval $[b_{min}, b_{max}] = [1.2, 9.2]$. This gives the discrete, true data function $n(b)$.
(ii) This true data function is disturbed with a relative error ϵ_r to get $f_i = n(b_i) \cdot (1 + \xi\epsilon_r)$, where ξ is a uniformly distributed random variable in $[-1, 1]$.
(iii) To each data point (b_i, f_i) we assign a standard deviation (the measurement error) σ_i, which we assume to be proportional to the relative error ϵ_r. The proportionality factor only scales scales the merit function χ^2, but does not change the minimizing parameters. Therefore, we simply set $\sigma_i = |f_i|$. We thus have the "experimental" data set (b_i, f_i, σ_i).
(iv) We fit the "experimental" data set (b_i, f_i, σ_i) by the polynomial $n^{(K)}(b)$ in b^{-1} of order K that minimizes the merit function (28).
(v) We analytically invert this polynomial with Eq. (19) to obtain the reconstructed source function $g^{(K)}$ which is an approximation of $g = 1/(rv(r))$.
(vi) From the reconstructed source function we obtain the reconstructed velocity law $v^{(K)}(r)$ and its absolute and relative standard deviations, evaluated for the N grid points $r_i = b_i$:

$$\sigma_a^2(v) = \frac{1}{N} \sum_{i=1}^{N} \left(v^{(K)}(r_i) - v(r_i)\right)^2 \quad , \tag{38}$$

and

$$\sigma_r^2(v) = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{v^{(K)}(r_i) - v(r_i)}{v(r_i)}\right)^2 \quad . \tag{39}$$

(vii) We repeat steps (ii) to (vi) one hundred times in order to get a statistical set of randomly disturbed data set, and we calculate the mean values $\overline{\sigma_a(v)}$ and $\overline{\sigma_r(v)}$ for the absolute and relative standard deviations defined by Eqs. (38) and (39).

Since in our application the source function is of little perspicuity we prefer to calculate the error for the corresponding reconstructed velocity law. This velocity law is more clearly represented on a linear scale. Thus, in addition to the relative standard deviation an estimate for the absolute uncertainty in the reconstructed velocity law might also be of interest. We therefore give in Table 1 and 2 some numerical results for the mean absolute and relative standard deviations $\overline{\sigma_a(v)}$ and $\overline{\sigma_r(v)}$.

Table 1. Mean standard deviations $\overline{\sigma_a(v)}$ (first value) and $\overline{\sigma_r(v)}$ (second entry) for the three reconstructed velocity laws given in the text. Note that the absolute standard deviations are given in units such that the terminal velocity is equal to 1. We show the behaviour of the inversion for $K = 4$ as a function of ϵ_r.

ϵ_r	$\beta = 0.5$	$\beta = 2.5$	cool giant
0.05	0.019 / 0.024	0.050 / 0.218	0.307 / 0.557
0.1	0.030 / 0.040	0.049 / 0.220	0.306 / 0.560
0.25	0.080 / 0.112	0.052 / 0.242	0.296 / 0.580
0.5	0.370 / 0.636	0.126 / 0.410	0.276 / 0.714
0.75	1.943 / 2.326	1.270 / 2.744	2.099 / 3.135
1.0	94.6 / 108.5	75.2 / 141.7	29.4 / 80.5

Table 2. Same as Table 1 but as a function of K for a fixed relative data error $\epsilon_r = 0.2$.

K	$\beta = 0.5$	$\beta = 2.5$	cool giant
4	0.060 / 0.084	0.049 / 0.231	0.300 / 0.570
6	0.155 / 0.264	0.068 / 0.122	0.332 / 0.360
8	0.244 / 0.444	0.057 / 0.107	0.254 / 0.280
10	0.223 / 0.418	0.097 / 0.162	0.312 / 0.340
12	0.281 / 0.550	0.083 / 0.145	0.415 / 0.443
15	0.812 / 1.570	0.074 / 0.136	0.165 / 0.189
20	0.625 / 1.223	0.106 / 0.223	0.149 / 0.173

Extensive numerical experiments showed that our inversion method works good for $\epsilon_r \leq 0.5$ (see Table 1). For bigger errors the noise amplification of the inversion becomes huge, since the inherent smoothing of our method is no longer effective. As in the usual inversion techniques a data smoothing is required prior to the inversion. As can be seen from Table 2 there is no straightforward correlation between the quality of the inversion and the adopted fitting order K. We can also not give a correlation between $\overline{\sigma_r(v)}$ and the data error ϵ_r. We just note that because of the increasing noise amplification a better fit to the "experimental" data by a higher order polynomial must not result in a better inversion.

7. Discussion

We have presented a new mathematical method for the inversion of Abel's integral which is suited for the solution of stellar wind problems. We give an explicit recipe for the application of this method. However, in every specific application our method most likely deserves some adaptation in order to account for the particular circumstances. Here, we summarize the simplifying assumptions which entered our analysis. First, in general the integration of the column density in Eq. (2) runs from $-\infty$ to the position of the compact star, or even more generally, to the boundary of the neutral wind region in the case where the compact star is able to ionize a fraction of the wind emanated by the companion. Second, the column density of absorbers might in reality be difficult to determine, because photons can be rescattered into the line of sight from the whole neutral part of the wind. Therefore, our assumption of pure absorption might only be valid for stellar winds that are thin enough, e.g. in the chromospheres of some late type giants. Last, the radiation source is in general not point–like, but has some finite extension which introduces additional difficulties like limb darkening effects. As an example for such a system and the related problems we refer to the eclipsing binary V 444 Cyg (WN 5 + O6) (Cherepashchuk 1974).

In spite of this more subtle considerations, it was shown by Vogel (1991) that all these simplifications are valid in the case of the symbiotic binary EG And. We therefore believe to be on the way for a more mathematical approach to empirically determine the wind structure of the cool giants in symbiotic systems like EG And, SY Mus, BF Cyg, and others.

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