

## Divergence and Curl of vector fields

In the two dimensional plane, a vector field  $F(x, y) = (P(x, y), Q(x, y))$  is defined by two functions  $P, Q$  of two variables. The curl of  $F$  is defined as  $\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y)$ . The divergence is defined as  $\text{div}(F)(x, y) = P_x(x, y) + Q_y(x, y)$ . Both curl and divergence are scalar fields. They are functions of two variables. If you think of  $F$  as a velocity field of a fluid, then  $\text{curl}(F)(x, y)$  measures the "rotation" of the fluid at a point and  $\text{div}(F)(x, y)$  measures the "expansion" of the fluid.

In three dimensional space, the vector field  $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$  is defined by three functions  $P, Q, R$  of three variables. The curl of  $F$  at a point  $(x, y, z)$  is defined as the vector  $\text{curl}(F)(x, y, z) = (R_y(x, y, z) - Q_z(x, y, z), P_z(x, y, z) - R_x(x, y, z), Q_x(x, y, z) - P_y(x, y, z))$ . The divergence of  $F$  at a point  $(x, y, z)$  is defined as  $\text{div}(F)(x, y, z) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z)$ . Note that now, the divergence is a scalar field and the curl is a vector field. The divergence again measures the "expansion" of the fluid at a point, the curls direction is the direction in which the rotation is maximal. The length of the curl vector measures the amount of rotation.

## Linear orthogonal transformations

Rotations are implemented as the composition of three rotations  $R = GBA$  where  $A$  is a rotation about the angle  $\alpha$  around the  $z$  axes, and  $B$  is a rotation about the angle  $\beta$  around the  $y$  axes and  $G$  is a rotation about the angle  $\gamma$  around the  $x$  axes.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & \sin(\gamma) \\ 0 & -\sin(\gamma) & \cos(\gamma) \end{bmatrix} \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can expand this to get

$$R \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos(\alpha) \cos(\beta)x + \cos(\beta) \sin(\alpha)y + \sin(\beta)z \\ (-\cos(\gamma) \sin(\alpha) - \cos(\alpha) \sin(\beta) \sin(\gamma))x + (\cos(\alpha) \cos(\gamma) - \sin(\alpha) \sin(\beta) \sin(\gamma))y + \cos(\beta) \sin(\gamma)z \\ (-\cos(\alpha) \cos(\gamma) \sin(\beta) + \sin(\alpha) \sin(\gamma))x + (-\cos(\gamma) \sin(\alpha) \sin(\beta) - \cos(\alpha) \sin(\gamma))y + \cos(\beta) \cos(\gamma)z \end{bmatrix}$$

While it would be more elegant to have this linear transformation implemented as a separate routine, we do it locally. The reason is that we do not want to compute the trigonometric functions every time new. Whenever possible, we store say  $\sin(\alpha)$  in a new variable and reuse it. These optimizations make the programming less elegant, but faster. Elegance and efficiency are somehow diametral.

## The wheel measures the curl in space

If we put a small wheel at a point  $(x, y, z)$  with an axes parallel to a vector  $v$ , then it spins with the amount  $\text{curl}(F)(x, y, z) \cdot v$ , where  $\vec{w} \cdot \vec{v}$  is the dot product between two vectors. A basic formula in vector calculus gives the wheel rotation  $\text{curl}(F)(x, y, z) \cdot v = |\text{curl}(F)(x, y, z)| \cdot |v| \cos(\theta)$ , where  $\theta$  is the angle between the curl vector and  $v$ .

In the programs, the user can change the vector  $v$  at a point and see the resulting wheel rotation. This allows to visualize measure the curl. Turn the vector until the wheel spins with maximal speed in the counterclockwise direction. This is the direction of the curl vector. The speed of the wheel is the length of the curl vector at this point.