Lecture 34: Calculus and Statistics

In this lecture, we look at an application of calculus to statistics. We have already defined the probability density function f called PDF and its anti-derivative, the cumulative distribution function CDF.

Probability density

Recall that a probability density function is a function f satisfying $\int f(x)\, dx = 1$ and which has the property that it is ≥ 0 everywhere. We say f is a probability density function on an interval $[a, b]$ if $\int_a^b f(x)\, dx = 1$ and $f(x) \geq 0$ there. In such a case, we assume that f is zero outside the interval.

Recall also that we called the antiderivative of f the cumulative distribution function $F(x)$ (CDF).

Expectation

The **expectation** of probability density function f is

$$m = \int_{-\infty}^{\infty} x f(x)\, dx \; .$$

In the case, when the probability density function is zero outside some interval, we have

The **expectation** of probability density function f defined on some interval $[a, b]$ is

$$m = \int_a^b x f(x)\, dx \; .$$

As the name tells, the expectation tells what is the average value we expect to get.

Variance and Standard deviation

The **variance** of probability density function f is

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx - m^2 \; ,$$

where m is the expectation.

Again, if the probability density function is defined on some interval $[a, b]$ then

The **variance** of probability density function f is

$$\int_a^b x^2 f(x)\, dx - m^2 \; ,$$

where m is the expectation of f.

The square root of the variance is called the **standard deviation**.

The standard deviation tells us what deviation we expect from the mean.

Examples

In the lecture, we will compute this in some examples. Here is some sample.

1 The expectation of the geometric distribution $f(x) = e^{-x}$

$$\int x e^{-x}\, dx = 1 \; .$$

The variance of the geometric distribution $f(x) = e^{-x}$ is 1 and the standard deviation 1 too.

Remember that we can compute also with Tic-Tac-Toe:

$$\int x^2 e^{-x}\, dx$$

x^2	e^{-x}	
$2x$	$-e^{-x}$	$\oplus$
2	e^{-x}	$\ominus$
0	e^{-x}	$\oplus$

2 The expectation of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\int_0^{\infty} x (2\pi)^{-1/2} e^{-x^2/2}\, dx = 0 \; .$$

3 The variance of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\int_0^{\infty} x (2\pi)^{-1/2} x^2 e^{-x^2/2}\, dx = 0 \; .$$

We can do that by partial integration too. Its a bit more tricky.

The next example is for trig substitution:

4 The distribution on $[-1, 1]$ with function $(1/\pi)(1-x^2)^{-1/2}$ is called the arcsin distribution. What is the cumulative distribution function? What is the mean m? What is the standard deviation σ? We will compute this in class. The answers are $m = 0, \sigma = 1/\sqrt{2}$.

Homework

1 The function $f(x) = \cos(x)/2$ on $[-\pi/2, \pi/2]$ is a probability density function. Its mean is 0. Find its variance

$$\int_{-\pi/2}^{\pi/2} x^2 \cos(x)\, dx \; .$$

2 The **uniform distribution on** $[a, b]$ is a distribution, where any real number between a and b is equally likely to occur. The probability density function is $f(x) = 1/(b - a)$ for $a \leq x \leq b$ and 0 elsewhere. Verify that $f(x)$ is a valid probability density function.

3 Verify that the function which is 0 for $x < 0$ and equal to

$$f(x) = \frac{1}{\log(2)} \frac{e^{-x}}{1 + e^{-x}}$$

for $x \geq 0$ is a probability density function.

4 A particular **Cauchy distribution** has the probability density

$$f(x) = \frac{1}{\pi} \frac{1}{(x-1)^2 + 1} \; .$$

Verify that $f(x)$ is a valid probability density function.

5 Find the cumulative distribution function (CDF) $F(x)$ of f in the previous problem.