BILLIARDS THAT SHARE A TRIANGULAR CAUSTIC

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ABSTRACT

We consider a one-parameter family of billiard tables T_ℓ which have as a common caustic the equilateral triangle γ. The billiard tables T_ℓ are constructed geometrically by the string construction, where the length ℓ of the string is the parameter. We study the family of circle homeomorphisms f_ℓ obtained by restricting the billiard map to the canonical invariant circle Γ_ℓ belonging to the caustic and the rotation function $\rho(\ell) = \rho(f_\ell)$. We show that the graph of ρ is a devil's staircase. We analyze the passage of a Birkhoff periodic orbits through the caustic as the parameter changes.

1 A Family of Billiard Maps

Let γ be the equilateral triangle with sidelength 1/3. We construct tables T obtained from γ by the so called *string construction* (see i.e. [11, 12]). One takes an unstretchable string having length $\ell > 1$, wraps it around γ, pulls it tight at a point, M, and drags it around γ. The point M then traces the table. Varying the parameter $\ell \in [1, \infty]$, we get a one-parameter family of billiard tables T_ℓ. Each of these tables is composed of piecewise elliptic arcs. (See Fig. 1.).

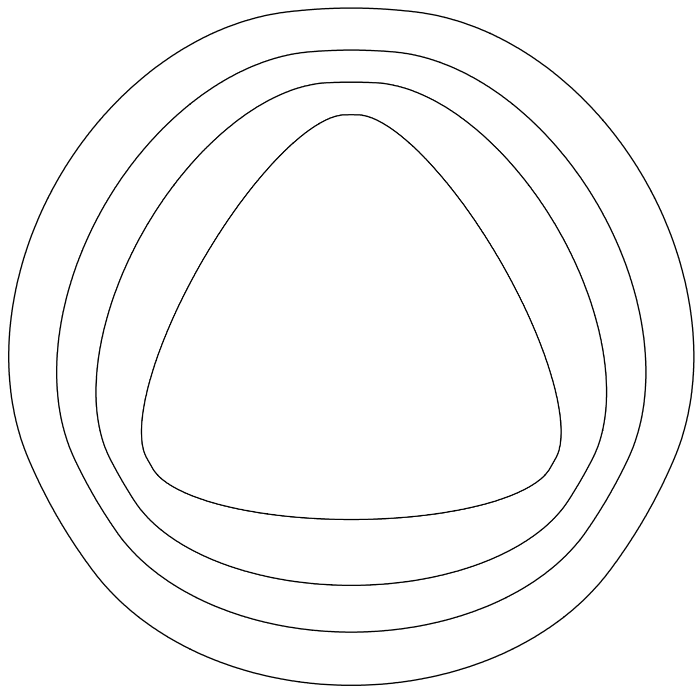
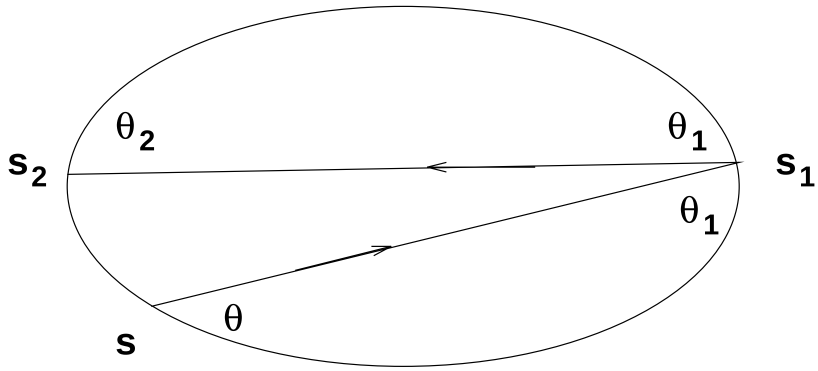


Fig. 1. Four examples of tables given by the string construction at an equilateral triangle.

The corresponding billiard maps, $\phi_\ell : \Phi \to \Phi$, form a natural one-parameter family of *twist maps*. We investigate this family in the present work.

We denote by Φ the space of unit vectors, with foot points in T_ℓ, directed inwards. The space Φ is a closed cylinder with natural coordinates (s, θ), where $0 \leq s \leq |T_\ell|$ is the normalized arc-length parameter on T, and where $0 \leq \theta \leq \pi$ measures the height in Φ. The cylinder Φ is the phase space for the billiard map, $\phi : \Phi \mapsto \Phi$, of the billiard table T (Fig. 2.).



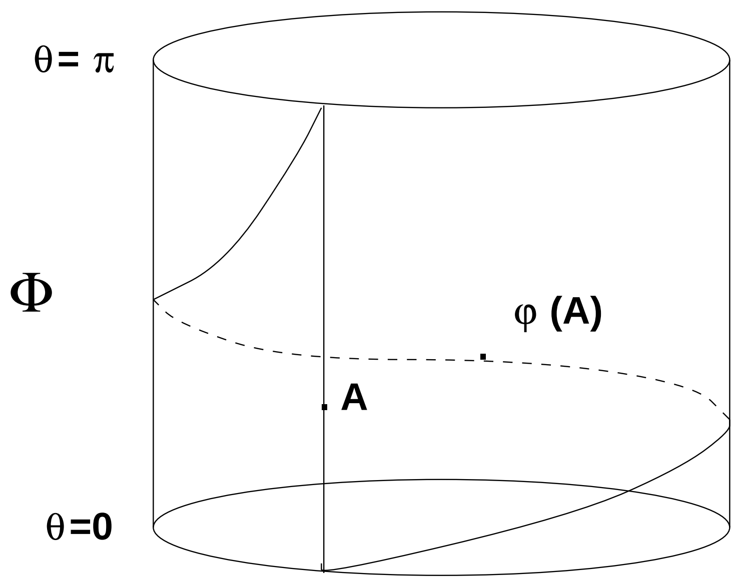
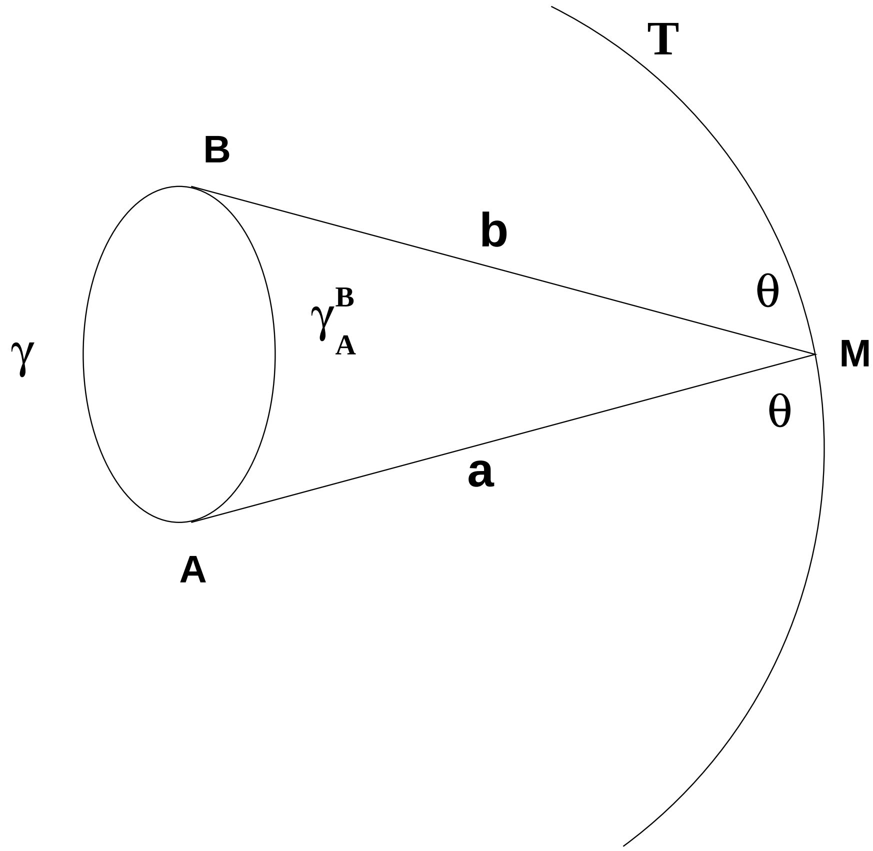


Fig. 1. Four examples of tables given by the string construction at **Fig. 2.** The billiard map ϕ and the phase space Φ.

By construction, each billiard map ϕ_ℓ has a canonical invariant circle, $\Gamma_\ell \subset \Phi_\ell$. It is formed by the rays supporting γ. Their orientation is induced by the positive orientation of γ. The opposite choice of orientation yields an other invariant circle $\tilde{\Gamma}_\ell$. The curve γ is the *caustic* corresponding to the invariant circle Γ_ℓ, for any ℓ [3].



f Fig. 3. A family T_ℓ of billiard tables obtained by the string construction. ($\ell_1 < \ell < \ell_2$). The invariant circles Γ_ℓ are moving up in the phase space Φ as ℓ increases.

Thus the family $T(\gamma, \ell)$, consists of billiard tables with the same caustic γ.

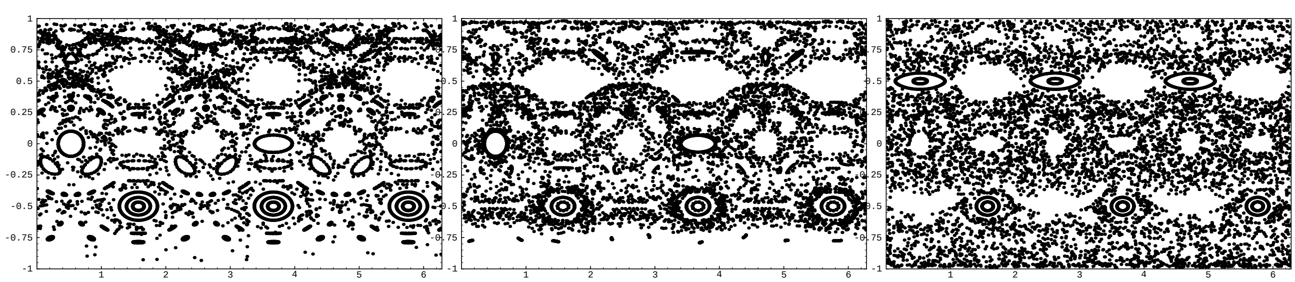


Fig. 4. Some orbits for the parameter $\ell = 1.25$.

Fig. 5. Some orbits for the parameter $\ell = 1.33$.

Fig. 6. Some orbits f5r the parameter $\ell = 1.38$.

By our choice of Γ_ℓ, the *rotation number* $\rho(\ell)$ of Γ_ℓ satisfies $\rho(\ell) \le 1/2$.

2 A Family of Circle Maps

We denote by $f_\ell : \Gamma_\ell \mapsto \Gamma_\ell$ the circle homeomorphism induced by ϕ_ℓ. We choose a reference direction in $\mathbb{R}^2$, and parameterize Γ_ℓ by the directions δ of the supporting rays. With this parameterization, f_ℓ becomes a family of Lipschitz homeomorphisms of the *standard circle* $S^1 = \mathbb{R}/\mathbb{Z}$. The rotation number $\rho(\ell) = \rho(f_\ell)$ is a continuous, nondecreasing function. This function $\rho(\ell)$ contains significant information about the family $\phi_\ell, |\gamma| < \ell$ of twist maps.

Lemma 2.1 *The map* $\rho \mapsto \rho(f(\ell))$ *is continuous and we have*

$$\lim_{\ell \to 1} \rho(\ell) = 1/3, \ \lim_{\ell \to \infty} \rho(\ell) = 1/2 \ .$$

Lemma 2.2 *Fix* $\ell > |\gamma|$*, and let* $f_\ell : S^1 \mapsto S^1, f_\ell(\delta) = \delta_1$*, be the corresponding circle homeomorphism. Denote by* $a = a(\delta), b = b(\delta)$ *the lengths of the supporting segments. (See the Fig. 8. for notation). Then* $f'_\ell(\delta) = d\delta_1/d\delta = a/b$*. The derivative with respect to* ℓ *is given by*

$$\frac{\partial f_\ell}{\partial \ell} = (u(B) + b \tan\theta)^{-1} \ .$$

Proof. See Fig. 7. and Fig. 8.

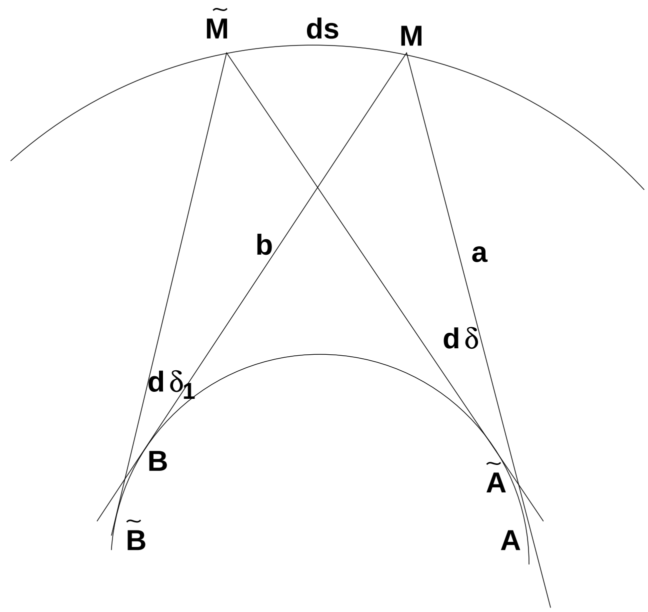


Fig. 7. Proof of the formula $f'_\ell(\delta) = d\delta_1/d\delta = a/b$.

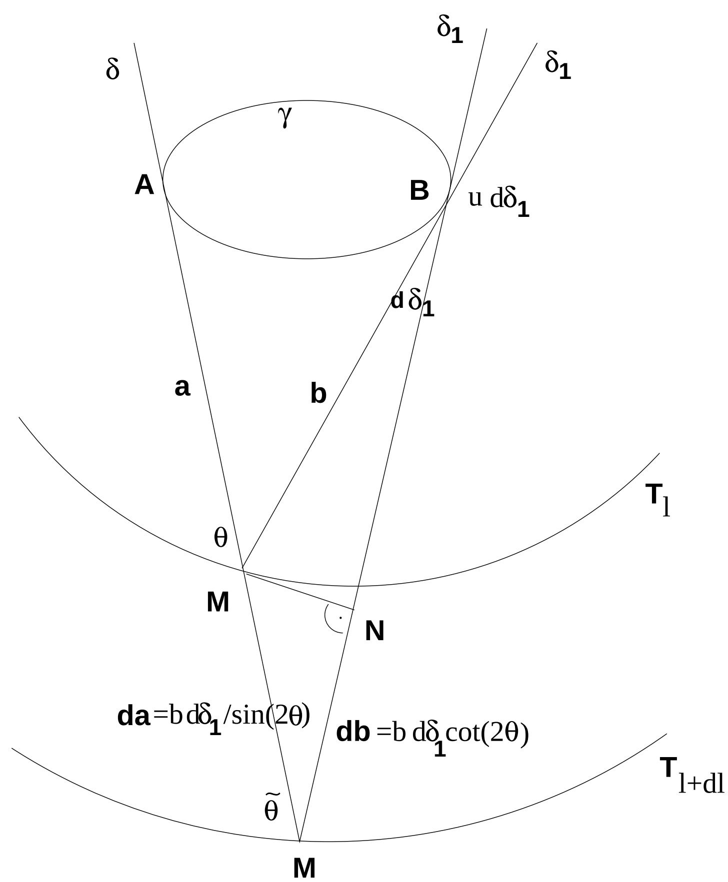


Fig. 8. Proof of the formula

$$\frac{\partial f_\ell}{\partial \ell} = (u(B) + b\tan\theta)^{-1} \ .$$

□

Corollary 2.3 *The one-sided derivatives $f'_\ell(\delta\pm)$ exist everywhere. The map $f(\ell)$ is C^1 outside a finite set of δ's.*

Recall that $t \in \mathbb{R}$ is a *point of increase* for a nondecreasing function, g, if $g(t-\epsilon) < g(t) < g(t+\epsilon)$ for all sufficiently small $\epsilon > 0$. Denote with $R(\alpha)$ the standard rotation by $2\pi\alpha$.

Proposition 2.4 *1. ρ is increasing at points ℓ for which $\rho(\ell)$ is irrational.*
2. ρ is increasing at points ℓ for which $\rho(\ell) = p/q$, and $f_\ell : S^1 \mapsto S^1$ is conjugate to the rotation $R(p/q)$ of S^1. In this case, f_ℓ^q is the identity on S^1.

Remark. Case 2. in this proposition, when $\rho(\ell) = p/q$ is especially important.

3 Devil's Staircase

The graph of a nondecreasing function, $\rho : (1,\infty) \mapsto (\rho(1), 1/2)$, is called a *devil's staircase* if there exists a family of disjoint open intervals, such that their union is dense in $(1,\infty)$, the function ρ is constant on each interval and such that ρ takes different values on different intervals (see, e.g., [7, 8]).

A rational number, $\rho(1) < p/q < 1/2$, is called *exceptional*, if the homeomorphism f_ℓ such that $\rho(\ell) = p/q$ is conjugate to $R_{p/q}$. For any *non-exceptional* p/q, the set $\rho^{-1}(p/q)$ is an interval, $I_{p/q} = [\ell^-_{p/q}, \ell^+_{p/q}]$ with nonempty interior. Using the terminology of circle maps, we call $I_{p/q}$ a *phase locking interval*. If p/q is exceptional, the interval $I_{p/q}$ degenerates to a single point $I_{p/q} = \{\ell_{p/q}\} = \rho^{-1}(p/q)$. In this case, every point on the invariant curve Γ_ℓ is periodic. From the formula in Lemma 2.2 of f_ℓ we get:

Lemma 3.1 *Let $\delta \in \Gamma_\ell$ be a periodic point, with period q. Let a_i, b_i, $1 \le i \le q$ be the lengths of the associated intervals in the polygon $P = P(\delta)$ (see Fig. 9.). Then the homeomorphism $f_\ell : S^1 \mapsto S^1$ is differentiable at the fixed point δ, and we have*

$$\frac{df_\ell^q(\delta)}{d\delta} = \frac{a_1 \cdots a_q}{b_1 \cdots b_q} \ .$$

This leads to the following Corollary:

Corollary 3.2 *1. If the curve Γ_ℓ contains an interval (δ', δ''), of periodic points of period q, then $(f_\ell^q)'(\delta) = 1$ on (δ', δ'').*

2. Γ_ℓ is p/q-exceptional if and only if

$$a_1 \cdots a_q = b_1 \cdots b_q \tag{1}$$

identically on Γ_ℓ.

By differentiating this relation along a deformation, we get necessary conditions which have to hold in an exceptional case.

Let $O \subset \Gamma_\ell$ be a q-periodic orbit, and let $O(t) \subset \Gamma_\ell, t \in I \subset \mathbf{R}$ be a *deformation* of O. We assume that I is an interval with a nonempty interior, that $0 \in I, O = O(0)$. Let $P(t) = M_1(t) \cdots M_q(t)$ be the polygon corresponding to $O(t)$. We will assume that the deformation $O(t)$ is differentiable for $t \in I$ and *nontrivial*. By this we mean that $t \to M_i(t)$ are C^1-functions, that $ds(M_i)/dt = ds_i/dt = \dot{s}_i \geq 0$ for $1 \leq i \leq q$, and that at least one $\dot{s}_j > 0$ for $t \in I$. We say a periodic orbit $O \subset \Gamma_\ell$ *admits a deformation* if O is contained in an interval of periodic orbits.

Lemma 3.3 *No periodic orbit $O \subset \Gamma_\ell$ admits a deformation.*

Proof. Assume a deformation $O(t)$ exists. Let $M_i(t)$ be parameterized by arc lengths $s_i(t)$. We have $\dot{a}_i = \cos(\theta_i)\dot{s}_i$ and $\dot{b}_i = -\cos(\theta_i)\dot{s}_i$. Use this and the mirror equation of geometrical optics to get

$$\dot{a}_i/a - \dot{b}_i/b = (1/a + 1/b)\cos(\theta_i)\dot{s}_i = 2\kappa(M_i)\cot(\theta_i)\dot{s}_i \; .$$

Differentiating $0 = \sum_i \log(a_i) - \log(b_i) = 0$ obtained from Equation (1) gives

$$0 = 2\sum_{i=1}^{q} \kappa(M_i)\cot\theta_i \dot{s}_i \; .$$

This is not possible since $\dot{s}_j > 0$ for at least one j and $\cot\theta_i > 0$ for all $1 \leq i \leq q$. $\square$

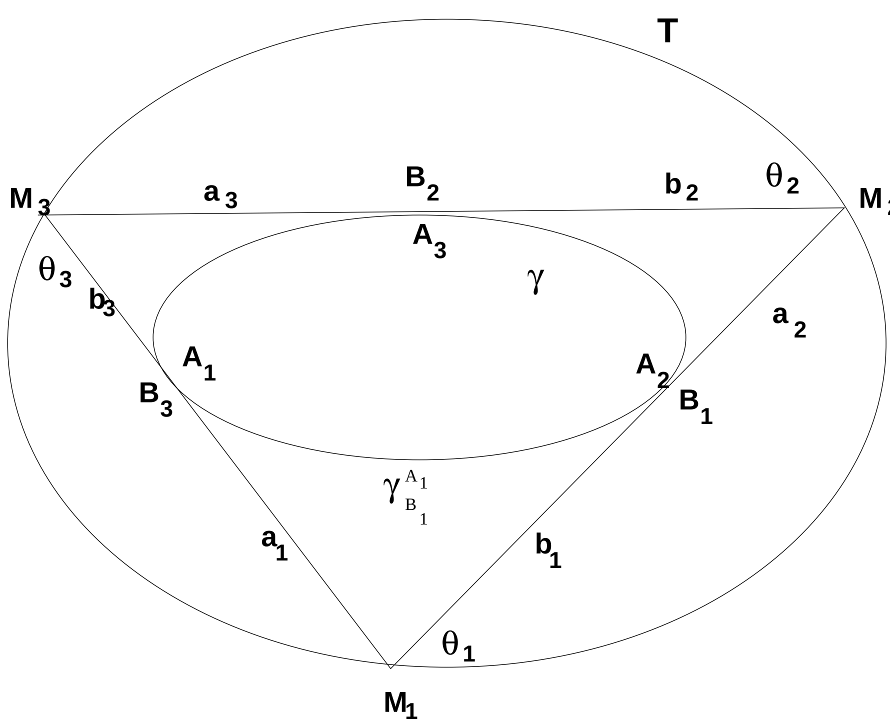


Fig. 9. Proof of Lemma 3.3.

Corollary 3.4 *No $p/q \in (1/3, 1/2)$ is exceptional. The function $\ell \mapsto \rho(f_\ell)$ is a devil's staircase.*

This is a special case of a more general theorem, which we have proven in [4]. There is first of all a geometrical argument which shows that for a generic convex curve γ, the rotation function is a devil's staircase. We also showed that if γ has a flat point of if γ is a polygon, then the rotation function is a devil's staircase.

We show next a numerically computed graph of the rotation function in our case, when the γ is an equilateral triangle. The graph shows the rotation number in dependence on ℓ.

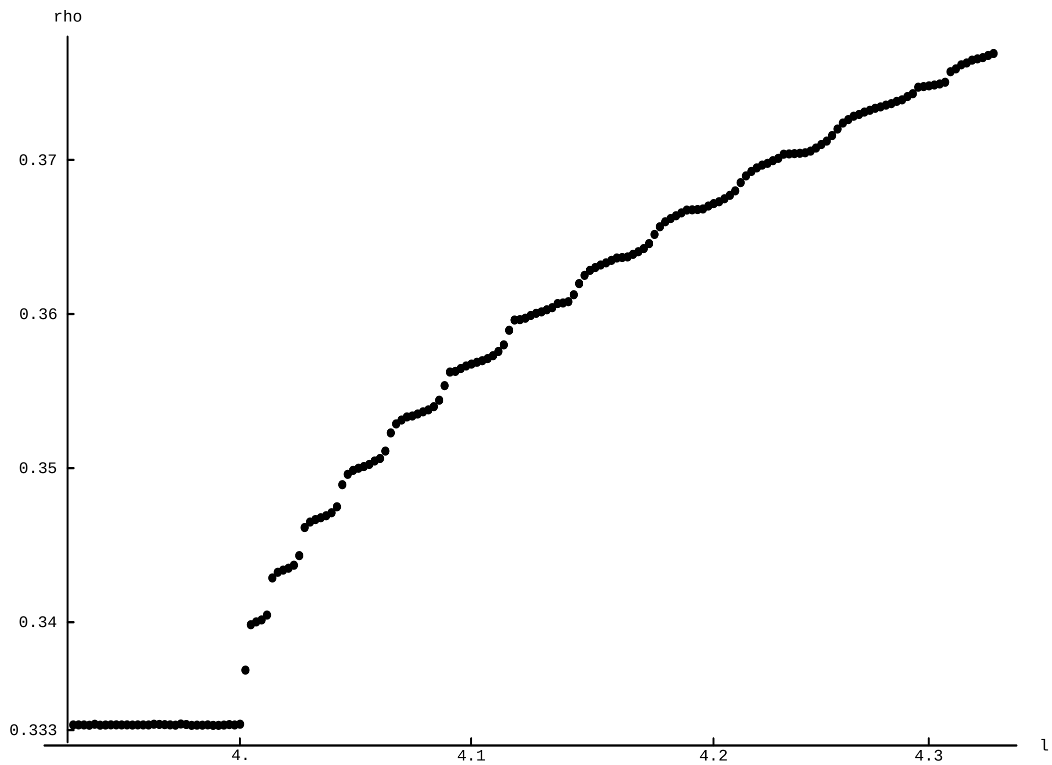


Fig. 10. Numerical computation of the rotation number in dependence of ℓ. The plateau at the beginning corresponds to rotation number 1/3. The end of the initial plateau is given by the value $\ell = 4/3$.

4 Aubry-Mather Sets and the Invariant Circle

In this section, we analyze the interaction of the moving invariant circle Γ_ℓ with Birkhoff periodic orbits of type p/q if the phase-locking interval $I_{p/q}$ has a nonempty interior.

It is a basic fact for monotone twist maps that for any $p/q \in [0,1]$ at least two Birkhoff periodic orbits of type (q,p) exist (see e.g. [2, 6]). In the billiard case, one of these orbits is a local maximum of the functional $|O|$, the total length of the trajectory O. Let $M_\ell(p/q)$ be the set of Birkhoff periodic orbits. Let α be irrational. An accumulation point (in the Hausdorff topology) of sets $M_\ell(p/q)$ as $p/q \to \alpha$ is called an *Aubry-Mather set* and denoted by $M_\ell(\alpha)$. Such a set $M_\ell(\alpha)$ has the property that it is the graph of a Lipschitz continuous function $\phi : K \subset S^1 \to [0,\pi]$ and that a lift $\tilde{\phi}$ of ϕ preserves the order of the covering of M [6].

(In the following discussion, we fix the rotation number $\alpha = p/q$. Typically, Birkhoff periodic orbits are isolated. The case of billiards shows however that one has to deal in general with whole arcs of Birkhoff periodic orbits If a connected set Y of periodic orbits contains a Birkhoff periodic orbit, then every orbit in Y is a Birkhoff periodic orbit and we will call such a set a *Birkhoff periodic set*.)

Let C be a simple, contractible closed curve in Φ avoiding all fixed points of ϕ^q. The *index* $\mathrm{ind}(C,\phi^q)$ of C with respect to ϕ^q is defined as the Brower degree of the map $v : C \to S^1$ $v(x) = (\phi^q(x)-x)/||\phi^q(x)-x||$. where $\phi^q(x) - x$ is calculated in a chart. The index is a homotopy invariant and does not change if we deform C without intersecting a fixed point of ϕ^q.

If a curve C contains only one fixed point x_0 of ϕ^q, $\mathrm{ind}(x_0,\phi^q) = \mathrm{ind}(C,\phi^q)$ is called the *index* of x_0. If C contains a connected fixed-point set Y_0, we call $\mathrm{ind}(Y_0,\phi^q) = \mathrm{ind}(C,\phi^q)$ the *index of this fixed point set*. If a curve C contains finitely many fixed points x_i (rsp. connected fixed-point sets Y_i) of ϕ^q, then $\sum_i \mathrm{ind}(x_i) = \mathrm{ind}(C,\phi^q)$.

We will use the fact that the index of a fixed point of an area-preserving homeomorphism is bounded above by 1 [10] [9].

Given a one-parameter family of monotone twist maps $\phi_\ell : \Phi \to \Phi$ parameterized by some interval I. Assume C is a simple closed curve in Φ such that for $\ell \in I$, no fixed point of ϕ_ℓ^q is on C and for all $\ell \in I$ only finitely many connected fixed point sets are inside C. A parameter value $\ell \in I$ for which the number of connected components of fixed points of ϕ_ℓ^q inside C changes is called a *bifurcation parameter*. Index considerations limit the possibilities for bifurcations of periodic orbits in monotone twist maps.

As the parameter ℓ varies, the the invariant circle Γ_ℓ moves through the phase space Φ. Sets in the region between the invariant circle and the boundary are called *below* Γ, the others are called *above* Γ.

For ℓ near $|\gamma| = 1$, the invariant circle is near the boundary $\{\theta = 0\}$ of the phase space Φ. For $\ell \to \infty$, the invariant circle Γ_ℓ moves towards the equator $E = \{\theta = \pi/2\}$ of Φ. Aubry-Mather sets $M_\ell(\alpha)$ with a fixed rotation number α pass through the moving circle Γ_ℓ.

The passage of $M_\ell(\alpha)$ with irrational α is easy to describe: since each irrational α is a point of increase of $\rho(\ell)$, the set of parameters ℓ for which $M_\ell(\alpha)$ intersects with Γ_ℓ consists of exactly one parameter value

$\ell_\alpha = \rho^{-1}(\alpha)$, for which $\Gamma_\ell = M_\ell(\alpha)$. The set $M_\ell(\alpha)$ is in general a Cantor set, for $\ell \neq \ell_\alpha$.

More interesting is when Birkhoff periodic orbits $M_{p/q}$ pass through Γ_ℓ if the phase locking interval $I_{p/q}$ is nontrivial. Since any periodic orbit on Γ_ℓ is a Birkhoff periodic orbit, $I_{p/q}$ is the set of parameters ℓ for which $M_{p/q}$ intersects with Γ_ℓ.

The following theorem is proven in [4] for a general convex caustic γ.

Theorem 4.1 *For $\ell \to \ell^-_{p/q}, \ell < \ell^-_{p/q}$, a Birkhoff periodic set of index -1 is approaching Γ_ℓ from above, intersects Γ_ℓ at $\ell = \ell^-_{p/q}$ with index -1 and leaves it for $\ell > \ell^-_{p/q}$ with index 1. For $\ell \to \ell^+_{p/q}$, $\ell < \ell^+_{p/q}$, a Birkhoff periodic set of index 1 approaches Γ_ℓ from above, hits it at $\ell = \ell^+_{p/q}$ and leaves it for $\ell > \ell^+_{p/q}$ with index -1. (see Fig. 11.)*

The idea of the proof is as follows: for $\ell = \ell^-_{p/q}$ and $\ell = \ell^+_{p/q}$, every Birkhoff periodic set on Γ_ℓ is parabolic and has index -1. By the symmetry of the situation, it suffices to study the bifurcation at ℓ^-. For $\ell < \ell^-$, there exists no Birkhoff periodic set on Γ_ℓ and all such sets are above Γ_ℓ. Local index conservation implies that such a set must exist nearby. Using properties of the one-parameter family of circle map f_ℓ for $\ell > \ell^-$ with ℓ near ℓ^-, we conclude that there exist 2 hyperbolic Birkhoff periodic sets of index -1 on Γ_ℓ. Local index conservation implies that a Birkhoff periodic set of index $+1$ must be nearby. By the global Poincaré index formula, we argue that this set V^- is on the other side of Γ_ℓ.

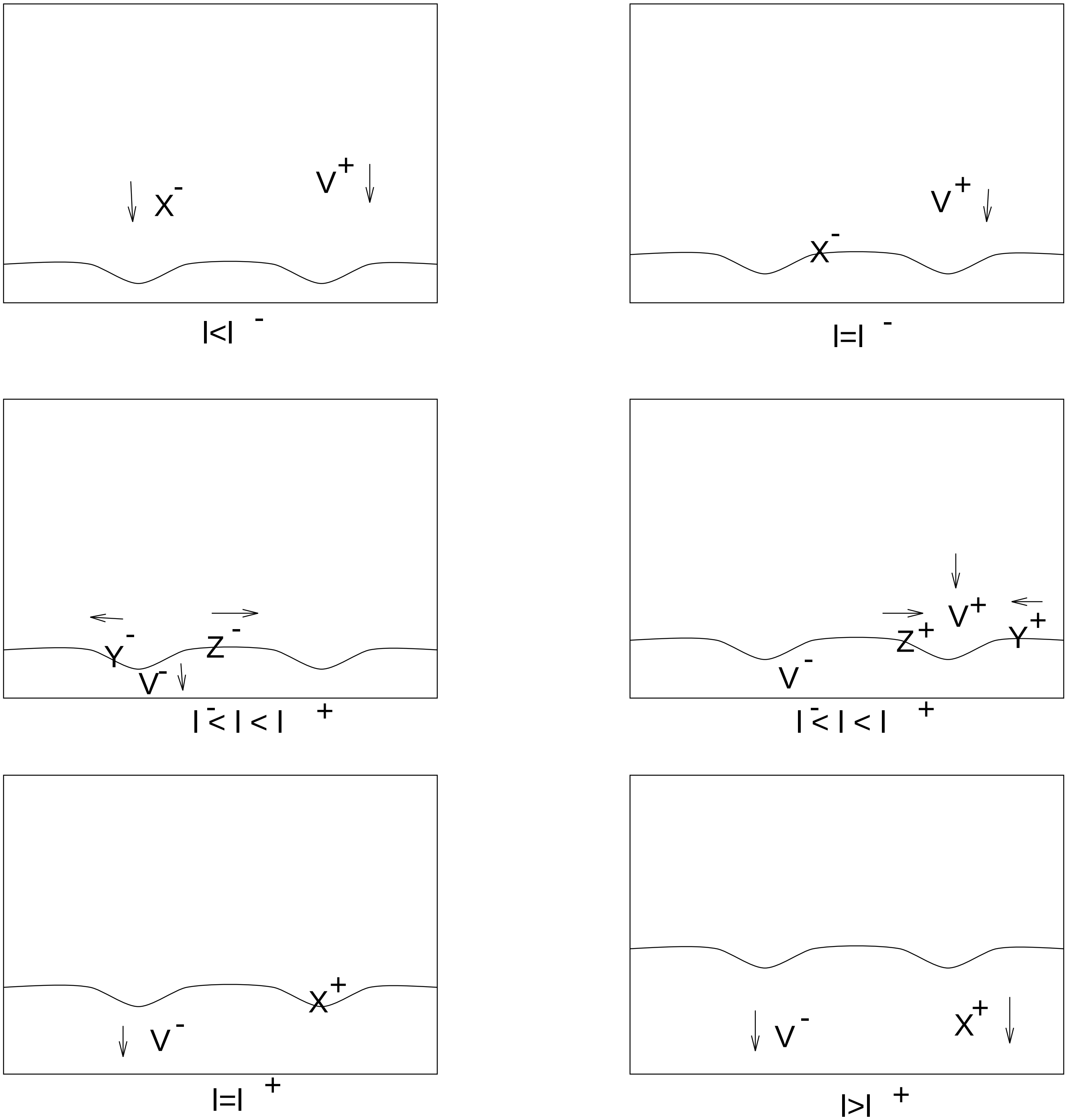


Fig. 11. Schematic illustration of the passage of the Birkhoff periodic orbits through the moving invariant curve Γ_ℓ.

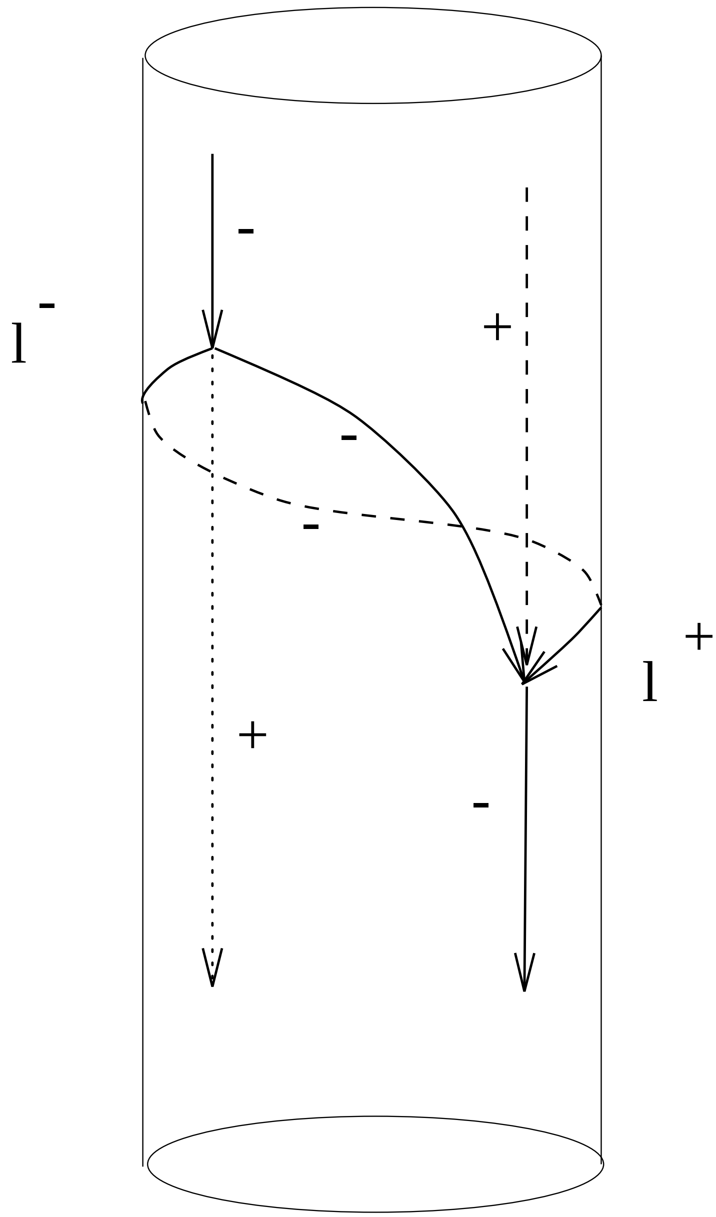


Fig. 12. Illustration of a possible path of the two Birkhoff periodic orbits in the phase space. While the parameter ℓ changes, both orbits head towards the invariant curve. The index -1 orbit hits the caustic, bifurcates to an index $+1$ orbit. The other remnants of the collision have index -1 and move along the caustic. When they meet again, the $+1$ orbit above Γ_ℓ merges with them to an index -1 orbit. After the bifurcation, the positive index set and negative index set move away from the invariant curve.

5 Discussion

Our numerical experiments provoke some open questions.

- We were especially interested in the parameter value $\ell = 4/3$, which is the supremum of all parameter values ℓ for which $\rho(\ell) = 1/3$. The numerical experiments do not decide clearly, if there exists a second caustics which is different from the canonical caustics Γ_ℓ.

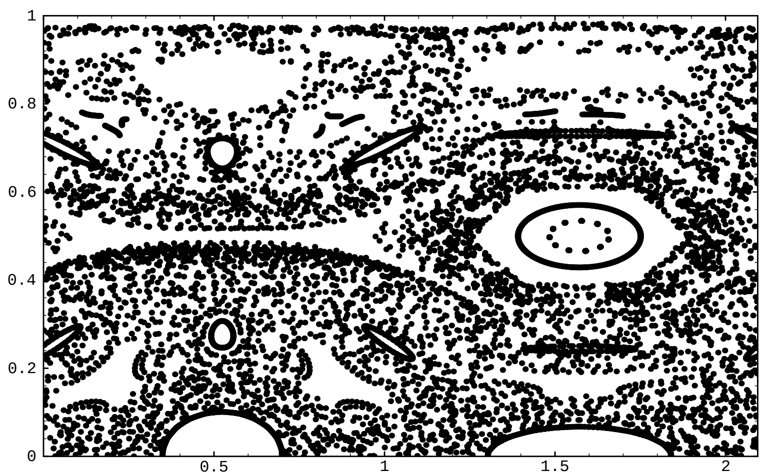


Fig. 13. The same picture as Fig. 5. Some orbits for $\ell = 1.33$. This time, we identified the dihedral symmetry of the phase space to see more details. We see the parabolic periodic orbit (see around coordinates $(0.5, 0.5)$) which is on the canonical invariant curve Γ_ℓ. On the same height to the right, there is an elliptic periodic orbit also above Γ_ℓ. The last invariant curve around that island has a hexagonal shape.

• It looks as if for ℓ in the interior of a phase locking interval, there exists a neighborhood of the canonical invariant circle Γ_ℓ, for which there exists no other invariant circle. One can argue that the stable or unstable manifold of the hyperbolic Birkhoff periodic orbits on the caustics prevent this if they do cross transversely outside the canonically invariant circle.

• The curvature of the tables T_ℓ has discontinuities. Hubacher's result [5] shows that near the boundaries of Φ, there exist no invariant curves. A result of Angenent [1] implies that for any $\ell > 1$, the topological entropy $h_{top}(\phi_\ell)$ of the billiard at T_ℓ is positive. It would be interesting to get quantitative results about the topological entropy. Is it monotonically decreasing in ℓ? Is $\limsup_{\ell \searrow 1} h_{top}(\phi_\ell) > 0$?

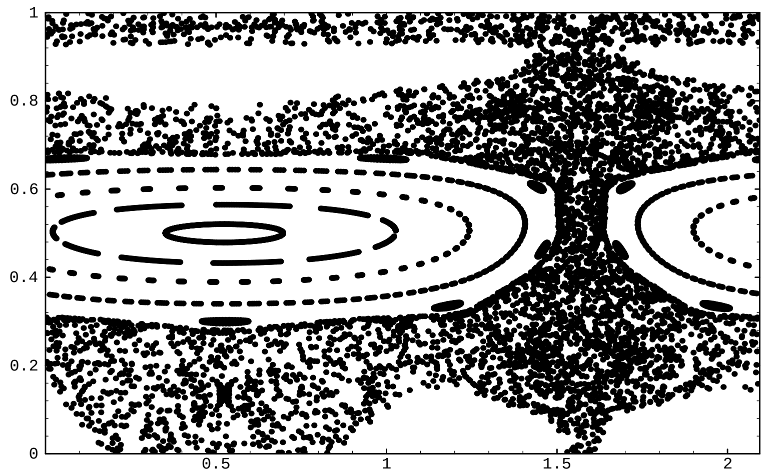


Fig. 14. Some orbits for $\ell = 1.00016$ for which the billiard table is very close to a triangle. Like in Fig. 12, we have identified the dihedral symmetry. (The arc length parameter displaying the first coordinate is slightly distorted since the program uses a convenient arc length instead of the real arc length).

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