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Weakly Mixing Invariant Tori of Hamiltonian Systems

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Abstract: We note that every finite or infinite dimensional real-analytic Hamiltonian system with a quasi-periodic invariant KAM torus of finite dimension $d \geq 2$ can be perturbed in such a way that the new real-analytic Hamiltonian system has a weakly mixing invariant torus of the same dimension.

1. Introduction

By the celebrated Kolmogorov–Arnold–Moser theory, Hamiltonian systems often have invariant tori on which the motion is quasi-periodic. While the dynamics on one-dimensional periodic orbits is always trivial, the Hamiltonian dynamics induced on higher-dimensional invariant tori can be interesting. The reason is the nontrivial ergodic theory of the systems

$$\frac{dx_i}{dt} = \alpha_i F(x)^{-1} \tag{1}$$

which are obtained by a change of time from a linear flow $\dot{x} = \alpha$ and which have the invariant measure $\mu = F(x)dx$ (see [1]). The flow ϕ^t can be either weakly mixing or can be conjugated to the linear flow $\dot{x} = \alpha$. Weak mixing means $\lim_{T\to\infty} T^{-1} \int_0^T \mu(Y \cap \phi^t(Y)) - \mu(Y)^2 dt = 0$ for any measurable set Y and is a weak type of chaos.

The question of what kind of dynamics occurs for a given α and F is interesting and has been studied for quite a while. Much is known in the case $d = 2$, a situation of wider interest because any smooth flow on the two-dimensional torus with no fixed points and some absolutely continuous invariant measure reduces to (1) by a change of coordinates [1]. In two dimensions and for smooth F, no strong mixing $\mu(Y \cap \phi^t(Y)) \to \mu(Y)^2$ can happen [4]. While Baire generically weakly mixing occurs [3], one can for almost all α conjugate the system to the quasi-periodic flow with $F = 1$. While for Diophantine rotation numbers, there is point spectrum by Kolmogorov's theorem and for Liouville rotation number, there is zero-dimensional but continuous spectrum in general [3], it is not known, if the dimension of spectral measures can become positive.

The ergodic theory of (1) is less understood in dimensions $d \geq 3$. We will note in Sect. 2 that near $F = 1$, we have generically weak mixing.

Arnold mentioned in an interview [6] that Kolmogorov's work on KAM theory was motivated by the question of whether mixing invariant tori exist for most Hamiltonian systems. While this question is open, our note shows that in a weak sense, the answer is yes: weakly mixing tori of dimension $d \geq 2$ occurs densely in some open sets of Hamiltonian systems. The result does not only apply to finite dimensional systems. KAM theorems for infinite dimensional Hamiltonian systems often lead to the persistence of finite-dimensional invariant tori [5,9]. In such a situation, one can perturb the infinite-dimensional Hamiltonian to obtain weakly mixing invariant measures of some PDE's.

2. A Higher Dimensional Version of Sklover's Theorem

Sklover has shown [10,1] that smooth differential equations on the two torus exist for which the dynamics is weakly mixing. The weak mixing property is even Baire generic [3] for real analytic F's. This can be generalized to higher dimensions:

Proposition 2.1 (Generalization of Sklover's theorem). *For a Baire generic set of (F, α) near $F = 1$, the flow $\dot{x}_i = \alpha/F(x)$ has purely singular continuous spectrum. Such systems are in general ergodic and weakly mixing.*

Proof. If the coordinates α_i of α are rationally dependent, then every orbit is periodic and $\mathbf{T}^n$ is foliated by one-dimensional tori, which are parameterized by $\mathbf{T}^{n-1}$.

A general measure preserving flow T_t on the torus $\mathbf{T}^n$ defines a one parameter family U_t of unitary operators $U_t f = f(T_{-t})$ on the Hilbert space $L^2(\mathbf{T}^d, dx)$. By Stone's theorem there is an infinitesimal generator L satisfying $U_t = \exp(iLt)$ which we call the Liouville operator.

There is a continuum of distinct ergodic invariant measures m_y and the Liouville operator L is an integral $L = \int_{\mathbf{T}^{n-1}} L(y)\ dy$, where $L(y) = p(y)\partial_x$ and $p(y)$ is the period of the flow. L has pure absolutely continuous spectrum on the orthocomplement of constant functions if and only if the measure of all orbits with a given period has measure zero. This condition for F is true for an open dense set of F's.

If α is Diophantine and the realanalytic F near 1, then the flow is conjugated to the linear flow and has pure point spectrum. This result of Arnold and Moser can also be derived from the fact that every time t map is conjugated to a map $x \mapsto x + \alpha$ (see [7, 2]).

There is a dense set of Liouville operators with absolutely continuous spectrum and a dense set of Liouville operators with discrete spectrum. By Simon's theorem [11], L Baire generically has purely singular continuous spectrum. (See [3] for details, like how to deal with the fact that the different Liouville operators are defined on different Hilbert spaces. Simon's theorem is: Let $\mathcal{X}$ be a complete metric space of self-adjoint operators on a separable Hilbert space for which convergence in the metric implies strong resolvent convergence. Suppose the two sets of operators in $\mathcal{X}$ that have purely continuous spectrum and purely discrete spectrum are both dense in $\mathcal{X}$. Then there is a dense G_δ in $\mathcal{X}$ of operators that have purely singular continuous spectrum.) □

Remark. While the result in two dimensions which we obtained together with A. Hof [3] is global, we don't know whether Proposition 2.1 can be made global in dimensions $d \geq 3$.

3. Weakly Mixing Invariant Tori

Consider a Hamiltonian vector field X_H with smooth Hamiltonian H on a symplectic manifold M. The function H is an integral of motion and the vector field X_H is tangential to any energy surface $C = \{x \mid H(x) = c\}$. If c is not a critical value for H, then C is a smooth submanifold and the vector field X_H does not vanish on C. If K is another Hamiltonian for which C is an energy surface,

$$C = \{x \mid H(x) = c\} = \{x \mid K(x) = c'\}$$

with $dH, dK \neq 0$ on C, then $\nabla K(x) = F(x)\nabla H(x)$ at every point $x \in C$ with $F(x) \neq 0$ and therefore

$$X_K(x) = F(x)X_H(x), \ \ F(x) \neq 0$$

on C. It follows that X_H and X_K have the same orbits on C although their time parameterization will be changed in general. Especially, any invariant torus of X_H which is contained in the energy surface C is an invariant torus of X_K and the change of the Hamiltonian produces a time change on this torus. Changing the Hamiltonian is useful for the study of periodic orbits (e.g. [12, Lemma 2.1]).

Lemma 3.1 (Poincaré trick). *Given a Hamiltonian system with a d-dimensional invariant torus N. For any smooth function F on N, with no roots on N, there exists a new Hamiltonian K which has the same invariant torus on which the dynamics is obtained by a change of time with function F.*

Proof. An explicit choice for K is $F(x)(H - c)$ which has $C = \{K = 0\}$. □

We also need to change the rotation vector on invariant quasi-periodic tori.

Lemma 3.2 (Change of the rotation vector). *Given a Hamiltonian system with a d-dimensional invariant torus N on which the dynamics is a rotation with rotation vector α. For any β near α, there exists a Hamiltonian K near H for which the torus N is still invariant and quasi-periodic with rotation vector β.*

Proof. A change of variables $A : z \mapsto (\phi, I)$ defined near the invariant torus N brings the Hamiltonian into action-angle variables on the invariant torus:

$$\dot{\phi} = \alpha + g(I, \phi), \dot{I} = h(I, \phi),$$

where $g(I, \phi)$ and $h(I, \phi)$ vanish on $N = \{I = \alpha\}$ (see e.g. [8]). If $\tilde{H}(\phi, I) = H(A^{-1}(\phi, I))$ is the Hamiltonian in these new variables, change it to $\tilde{K}(\phi, I) = \tilde{H}(\phi, I) + (\beta - \alpha)I$ and define $K(x, y) = \tilde{H}(A(x, y))$. The flow of X_K leaves N invariant and is conjugated there to a quasi-periodic flow with rotation vector β. □

Remark. The invariant torus N obtained like this loses the KAM property during the perturbation. However, by KAM, a different torus with the same rotation vector α will persist and the perturbed system will now have two invariant tori, one with rotation number α and one with rotation vector β.

Theorem 3.3. *Given a real-analytic Hamiltonian H for which there exists an invariant torus N on which the dynamics is quasi-periodic, there exists a Hamiltonian K arbitrarily close to H for which the same torus N is still invariant and for which the dynamics on N is ergodic and weakly mixing.*

Proof. Using Proposition 2.1, take (β, F) with β near α and F near 1 such that the corresponding flow on the torus is weakly mixing. We make now a first change of the Hamiltonian $H \to K_1$ such that the rotation vector of N is changed to β.

Let c be the energy of an orbit on N. Let $K = F(K_1 - c)$ be the Hamiltonian obtained from K_1. The flow of this Hamiltonian induced on the invariant torus N is weakly mixing. □

The result generalizes obviously to infinite-dimensional Hamiltonian systems for which KAM theory assures that finite dimensional tori survive (see [5]). For example, there are perturbations of some nonlinear wave equations which have weakly mixing invariant tori.

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