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Iteration of the coupled map lattice construction

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Abstract

We define a class $\mathcal{F}$ of topological dynamical systems, which are left invariant by coupled map lattice constructions. This class $\mathcal{F}$ has the property that if the coupling of the systems is sufficiently weak and (M, f) contains a hyperbolic set, then the new system $\phi(M, f)$ obtained by the coupled map lattice construction has a hyperbolic set too. The coupled map lattice construction, map ϕ, can be iterated and leads to dynamical systems (M_n, f_n) with a hierarchical diffusion structure. We obtain examples where shifts are embedded in all scales of the limiting system $(M_\infty, f_\infty) \in \mathcal{F}$.

1. Introduction

Since *coupled map lattice* (CML) theory was introduced by Kaneko in the early 1980s [1], the interest for such spatial extended systems has expanded into many fields. Originally CMLs were introduced as models for studying spatio-temporal chaos. Today, CMLs are used as models for many biological, chemical, physical and engineering problems (see, e.g., Refs. [2,3] and references therein).

A CML is a construction of dynamical cells distributed on some spatial lattice or graph. Each cell has a *local* dynamics given by some nonlinear discrete time dynamical system (M, f) that is coupled to the other cells through some coupling function g. A number of papers have been dealing with qualitative and numerical aspects of CMLs. For examples, we refer to Ref. [4].

The physical motivation for the CML construction comes from the need to model a variety of complicated behavior observed in experiments. The most challenging one is the problem of turbulence or spatio-temporal chaos that is seen for example in the Bénard convection. Until recently, there was no deterministic model of an extended system, where one had succeeded in proving that spatio-temporal chaos exists. However, Bunimovich and Sinai proved in Ref. [5] that for a CML consisting of a one-dimensional map with strong statistical properties and with weak space interaction, spatio-temporal chaos exists. Also for weakly coupled hyperbolic maps, this is shown to be true by Gundlach and Rand in Refs. [6–8].

In this article, we iterate the coupled map lattice construction. We start with the hypothesis that a complex system like for example a fluid or the human brain is organized in a hierarchical manner: on its finest scale, the system is described by a dynamical system (M, f). An approximation of a more global picture is to take a discrete set G_1 of such systems and to couple the systems with a coupling function g_1. This leads to a new and in general already infinite-dimensional system (M_1, f_1) which is a new (and hopefully better) approximation of the system. Take now a new arrangement G_2 of systems (M_1, f_1) and couple them with

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a function g_2 which gives a system (M_2, f_2). Iteration of this procedure constructs a sequence of models (M_i, f_i) each having a multiscale dynamics. If the g_i are linear, we could call it a multiscale diffusion.

We will show that if the coupling functions g_i are close enough to the identity, the limiting system (M_∞, f_∞), the ultimate model of the system, has hyperbolic behavior on infinitely many scales. This limiting system is again a topological dynamical system. If we fix M and G_i, then (M_∞, f_∞) depends continuously on the initial map f as well as the coupling functions g_i. The group $G_1 \times G_2 \times \ldots$ acts in a natural way on (M_∞, f_∞) and we will prove that under suitable conditions, there exist f_∞ invariant measures on M_∞ which are also $G_1 \times G_2 \times \ldots$ invariant and ergodic with respect to each of the G_i actions.

2. The coupled map lattice construction

Let X be a Banach space and let Γ be a discrete subgroup of X. In the following we assume that M is a manifold either given by (i) $M = X/\Gamma$ where M is compact in some weak metrizable topology, or that (ii) M is a compactification of X in some weak topology for which every closed ball is compact and such that the compactification is a Banach manifold.

We assume that the local dynamics is given by a differentiable map f: $M \to M$ which is continuous in the weaker topology (note that the differentiability and so continuity in the strong topology does not necessarily imply the continuity in the weaker topology). Denote by $\mathcal{F}$ the class of such topological dynamical systems (M, f) and let $\overline{\mathcal{F}}$ be the subset of $\mathcal{F}$ for which f is invertible and $(M, f^{-1}) \in \mathcal{F}$.

In order to see that the definition is reasonable, we give some examples, where M is of type (i) or type (ii). For type (i), we note that every differentiable map f on the d-dimensional torus $M = \mathbb{R}^d/\mathbb{Z}^d$ belongs to $\mathcal{F}$. Another example is constructed by letting $N = \mathbb{R}^k/\mathbb{Z}^k$ and defining $M = N^{\mathbb{Z}} = X/\Gamma$. Then M is a manifold over the Banach space $X = l^\infty(\mathbb{Z}, \mathbb{R}^k)$ with discrete additive subgroup $\Gamma = l^\infty(\mathbb{Z}, \mathbb{Z}^k)$. The weaker topology is the product topology. Any differentiable map $f(x)_n = h(x_{n-1}, x_n, x_{n+1})$ such that h is periodic in each variable is an example such that $(M, f) \in \mathcal{F}$.

For examples of type (ii), we note that every polynomial map f on $\mathbb{C}$ belongs to $\mathcal{F}$. If we extend f to $M = \mathbb{C}^* = \mathbb{C} \cup \infty$, then this Riemann sphere is a compact manifold and f induces a continuous map on the compact metric space M and is in the $1/z$ coordinates also differentiable in ∞. Another example is constructed by letting $N = \mathbb{C}^*$ and $M = (\mathbb{C}^*)^{\mathbb{Z}}$. Then M is a compactification of the Banach space $X = l^\infty(\mathbb{Z}, \mathbb{C})$ with the weak-* topology. Letting $f(x)_n = h(x_{n-1}, x_n, x_{n+1})$, where h is a polynomial in three variables as before, we get that (M, f) is in $\mathcal{F}$.

We now define a map $\phi(M, f) = (\tilde{M}, \tilde{f})$ which corresponds to a description of the system on the next hierarchical level. Given a discrete symmetry group G, let g be a (Fréchet) differentiable coupling function acting on M^G. We assume that the coupling is short ranged in the sense that $g(x)_n$ depends only on finitely many x_k. We assume further that g commutes with the shift action of G on M^G. Define for $(M, f) \in \mathcal{F}$

$$\phi(M, f) = (\tilde{M}, \tilde{f}),$$

where in case (i) $\tilde{M} = \tilde{X}/\tilde{\Gamma} = l^\infty(G, X)/l^\infty(G, \Gamma)$ and

$$\tilde{f}(x) = g(f^G)(x),$$

with $f^G(x)_n = f(x_n)$ for $n \in G$. The weaker topology on $\tilde{M}$ is the product topology. In the type (ii) case, $\tilde{M} = M^G$ is also a Banach manifold. If $(M, f) \in \mathcal{F}$, then also the new system $(\tilde{M}, \tilde{f})$ is in $\mathcal{F}$.

If $(M, f) \in \overline{\mathcal{F}}$, we define a different map $\overline{\phi}$ which has the property that $\overline{\phi}(M, f)$ is also in this class,

$$\overline{\phi}(M, f) = (\tilde{M}, \tilde{f}),$$

where $\tilde{M} = (M \times M)^G$ and

$$\tilde{f}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -f(y_n) + g(f(x_n)) \\ f(x_n) \end{pmatrix} = \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}.$$

The map $\overline{\phi}$ leaves the class $\overline{\mathcal{F}}$ of invertible dynamical systems invariant since we can invert ϕ,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} f^{-1}(\tilde{y}_n) \\ f^{-1}(g(\tilde{y}_n) - \tilde{x}_n) \end{pmatrix}.$$

Remarks. (1) The requirement that G is a group is sometimes not necessary and G can then be taken to be a discrete set. It is however natural to consider only groups since they can serve as symmetries of the systems under study.

(2) The map g could be generalized in various ways. We could for example allow that g is a function of $f, f^2, f^3, \dots, f^n$ as long as the differentiability condition remains true.

Examples. (1) If g is the identity, then we constructed a network of dynamical systems (M, f) which are evolving independently.

(2) If g is the nearest neighbor Laplacian on a Cayley graph of G, then $\tilde{f}$ describes a diffusion process on this graph.

(3) If G is the trivial group with one element, then g is a differentiable map on M. The iteration of ϕ is then equivalent with iterating g on M. One would therefore lose nothing by assuming that the initial dynamical system (M, f) is the identity map.

(4) Let the discrete group G have finitely many generators $g_1, \dots, g_d$ and let Δ_d be the nearest neighbor Laplacian on $X^G \to X^G$: $\Delta_d(x)_n = \sum_{|m-n|=1} x_m$, where $|m-n|$ is the distance in the Cayley graph of G. Take $g(x) = \epsilon \Delta_d(x)$. This is the classical coupled map lattice.

(5) An example of a simple nonlinear map ϕ: take $G = \mathbb{Z}$ and $g(x)_n = x_n(x_{n+1} + x_{n-1})$.

(6) An example motivated by the theory of discrete Hamiltonian systems: take $G = \mathbb{Z}$ and $g(x)_n = x_{n+1} - 2x_n + x_{n-1} + \gamma \sin(x_n)$, where γ is an additional parameter.

(7) If G is a finite group and (M, f) is finite dimensional, then also $\phi(M, f)$ is finite dimensional.

3. Hyperbolic sets and embedded shifts

A dynamical system (M, f) in $\mathcal{F}$ has a *shift embedded*, if there exists an f-invariant subset Y of M which is closed in the weaker topology, such that the topological dynamical system (Y, f) is topologically conjugated to the topological dynamical system $(A^{\mathbb{N}}, \sigma)$, where σ is the shift $\sigma(x)_n = x_{n+1}$ on $A^{\mathbb{N}}$ and where A is a finite alphabet. (M, f) has a p-dimensional shift embedded, if there exists an f-invariant subset Y of M and p commuting differentiable maps $f = f_1, f_2, \dots, f_p$, such that Y is f_i-invariant and $(Y, f_1, \dots, f_p)$ is topologically conjugated to the p-dimensional shift $(A^{\mathbb{N}^p}, \sigma_1, \dots, \sigma_p)$, where $\sigma_k(x)_n = x_{n+e_k}$. If the dynamical system (M, f) is in the class $\mathcal{F}$ of invertible systems, then we can also require that the shifts are invertible σ_i: $A^{\mathbb{Z}^p} \to A^{\mathbb{Z}^p}$.

An f-invariant closed subset Y of M is called a *hyperbolic set*, if df is a hyperbolic map on the tangent bundle E over Y: there exists $r < 1$ and a coinvariant splitting $E = E^+ \oplus E^-$ of the tangent bundle E such that $df(x)$: $E^+(x) \to E^+(fx)$ has the spectrum in $\{z \geqslant r^{-1}\}$ and $df(x)$: $E^-(x) \to E^-(fx)$ has the spectrum in $\{z \leqslant r\}$. For hyperbolic dynamics see, for example, Ref. [9].

Proposition 3.1. Assume $(M, f) \in \mathcal{F}$ has a hyperbolic set Y which satisfies in the type (ii) case: $Y \subset X \subset M$. If g is close enough to the identity map in $C(M^G, M^G)$, then $(\tilde{M}, \tilde{f}) = \phi(M, f)$ contains a hyperbolic set $\tilde{Y}$ and the dynamical system $(\tilde{Y}, \tilde{f})$ is topologically conjugated to (Y^G, f^G). The same is true for $\overline{\phi}$ on $\overline{\mathcal{F}}$.

Proof. We consider first the case $(M, f) \in \mathcal{F}$.

Denote by (Z, T) the topological dynamical system (Y^G, f^G). We construct the conjugation ϕ: $Z \to \tilde{M}$ by using the implicit function theorem. Consider the Banach space $C(Z, \tilde{X})$ with norm $||\phi||_\infty = \sup_{z \in Z} ||\phi(z)||$, where $||\ ||$ is the norm in the Banach space $\tilde{X}$. Define a map F: $C(Z, \tilde{X}) \times C^1(M^G, X^G) \to C(Z, \tilde{X})$ by

$$F(\phi, g)(z) = \phi(Tz) - g f^G \phi(z).$$

Then F is Fréchet differentiable. If ϕ satisfies $F(\phi, g) = 0$, then ϕ is a conjugation of T with $\tilde{f}$ since $\phi \circ T = \tilde{f} \circ \phi$. For $g = \mathrm{Id}$, the equation $F(\phi, \mathrm{Id}) = 0$ has as a solution the identity map $\phi_0(x) = x$. The functional derivative $\partial_\phi F(\phi_0, \mathrm{Id})$ is given by $U - df^G$, where U is the linear map $\phi \mapsto \phi(T)$ which has the spectrum on the unit circle. Since df was hyperbolic, also df^G is hyperbolic and $\partial_\phi F(\phi_0, \mathrm{Id})$ is invertible. The implicit function theorem (see, for example, Ref. [10]) assures that we get a solution ϕ for g close enough to the identity. Since ϕ is close to the identity map, it is still injective. The set $\tilde{Y} = \phi(Z)$ is the hyperbolic set for $\tilde{f}$.

The proof is similar if $\overline{\phi}$ acts on $\overline{\mathcal{F}}$: write $\tilde{X} = \tilde{X}_1 \oplus \tilde{X}_2$ and $\phi = (\phi_1, \phi_2)$. Take then

$$F\left(\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, g\right)(z) = \begin{pmatrix} \phi_1(Tz) + f^G(\phi_2(z)) - g(f^G(\phi_1(z))) \\ \phi_2(Tz) - f^G(\phi_1(z)) \end{pmatrix}.$$

Since $\phi \mapsto (f^G(\phi_2), f^G(\phi_1))$ is hyperbolic and $\phi \mapsto (\phi_1(T), \phi_2(T))$ has the spectrum on the unit circle, we obtain as before that $\partial_\phi F(\phi, \mathrm{Id})$ is invertible. The rest of the argument is the same. □

The system (M, f) has a *hyperbolic shift embedded*, if there exists a hyperbolic set Y, such that f restricted to Y is topologically conjugated to a shift. The system (M, f) is said to have a p-dimensional hyperbolic shift embedded on Y, if (Y, f) is a hyperbolic set and if there exist p commuting maps f_i which leave Y invariant such that $(Y, f_1, \dots, f_p)$ is topologically conjugated to a p-dimensional shift.

Corollary 3.2. Assume (M, f) has a k-dimensional hyperbolic shift embedded. If G contains the subgroup $\mathbb{Z}^d$ and if g is close enough to the identity, then $(\tilde{M}, \tilde{f}) = \phi(M, f)$ has a $(k + d)$-dimensional shift embedded.

Proof. If (Y, f) has a k-dimensional shift embedded, then $(Z, T) = (Y^G, f^G)$ has a $(k + d)$-dimensional shift $(T_1, \dots, T_{k+d})$ embedded. Define a map F: $C(Z, \tilde{X}) \times C(M^G, M^G) \to C(Z, \tilde{X})^{k+d}$ by

$$F(\phi, g)(z)_i = \phi(T_i z) - g(f^G \phi(z)).$$

Again $\partial_\phi F(\phi, \mathrm{Id})(z)$ is nonvanishing and as above, the implicit function theorem gives the conjugation for $||g - \mathrm{Id}||_\infty$ sufficiently small. □

Remark. Having a higher dimensional shift embedded implies "space-time-chaos" or "spatial-temporal chaos" in the sense that there exists a measure on $\tilde{M} = M^G$ which is both shift invariant and $\tilde{f}$ invariant and which is strongly mixing.

Corollary 3.3. If g is close enough to the identity, there exists an $\tilde{f}$- and G-invariant measure μ, which is ergodic. If G is infinite, μ is strongly mixing and has exponential decay of correlations.

Proof. This is a trivial consequence since the embedded system is measure theoretically isomorphic to a p-dimensional Bernoulli shift with $p \geqslant 2$. □

Examples. (1) Let $M = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be a one-dimensional circle and assume that f is the *standard circle map*,

$$f_\gamma(x) = x + \alpha + \gamma \sin(2\pi x).$$

Let $G = \mathbb{Z}$ and $g = \epsilon\Delta$ such that the new system is a map $\tilde{f}$ on the infinite dimensional torus given by

$$\tilde{f}(x)_n = f_\gamma(x_n) + \epsilon\Delta f_\gamma(x)_n \pmod 1.$$

As an illustration, we give a direct proof that the map $\tilde{f}$ has a two-dimensional shift embedded even without assuming that f has this property. Define $\epsilon' = 1/\gamma$ and consider the two-dimensional shift (Z, T_1, T_2) with $Z = \{0,1\}^{\mathbb{Z}^2}$. We want to find a map ϕ: $Z = \{0,1\}^{\mathbb{Z}^2} \to \mathbb{T}^{\mathbb{Z}}$ such that $\phi(T_1 z)_n = (U\phi(z))_n$ and $\phi(T_2 z)_n = \tilde{f}(\phi(z)_n)$. Define $h_\delta(q) = \delta(q + \alpha) + \sin(q)$ and a map F: $\mathbb{R}^2 \times C(X) \to C(X)$,

$$F(q, \epsilon, \delta) = \delta q(T_1) - h_\delta(q) - \epsilon\delta\Delta h_\delta(q).$$

If $F(q, \epsilon, \delta) = 0$, then ϕ: $X \to Y, \phi(x) = \{q(T_1^n x)\}_{n\in\mathbb{Z}} \in Y = M^{\mathbb{Z}}$ conjugates the system (X, T_1, T_2) with $(K, \tilde{f}, U)$, where $K = \phi(X)$. The fact $\phi \circ \sigma_1(x) = U \circ \phi(x)$ is clear. The conjugation $\phi \circ \sigma_2(x) = \tilde{f} \circ \phi(x)$ follows from $F(q, \epsilon, \delta) = 0$. For $q(x) = x_0$, we get

$$F(q, 0, 0) = \sin[q(x)] = 0$$

and $\partial_q F(q, 0, 0)u = \cos(q)u$ is invertible. The implicit function theorem gives a solution q for ϵ and δ sufficiently small.

(2) Take the quadratic map f: $\mathbb{C} \mapsto \mathbb{C}, z \mapsto x^2 + \gamma$ for $|\gamma| > 2$. This map has the Julia set as the hyperbolic set. Take $M = \mathbb{C}^*$, the Riemann sphere. Interesting is already the case of a finite cyclic group $G = \mathbb{Z}_n$ and where $g(x) = \epsilon\Delta x$ is linear. In this case, the new map $\phi(M, f)$ is an analytic map on $\mathbb{C}^n$. Analogous to (1), we can directly show the existence of a shift for large γ, without knowing that f has one.

(3) Take an invertible system (M, f), where $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and where f is the standard map f_γ: $M \to M$,

$$f(x, y) = (2x - y + \gamma \sin(2\pi x), x),$$

which is invertible and which has a lot of hyperbolic invariant sets for $|\gamma|$ sufficiently large. If G is finite, the map $\overline{\phi}$ gives area-preserving maps in $\mathbb{R}^{4n}$ which have hyperbolic sets.

4. The limiting system

Let (M_i, f_i) be the sequence of dynamical systems obtained by iterating the coupled map lattice construction using a sequence g_i of maps on $M_{i-1}^{G_i}$. Define a limiting system (M_∞, f_∞) as follows. Take the product system $(M_1 \times M_2 \times \ldots, f_1 \times f_2 \times \ldots)$ which is a topological dynamical system in $\mathcal{F}$. Define an equivalence relation $\sim$ on this product space by saying $x \sim y$ if $x_n = y_n$ for sufficiently large n. Since the map $f_1 \times f_2 \times \ldots$ respects this equivalence relation, we can pass to the quotient system which is again a topological dynamical system. Denote this system by (M_∞, f_∞) and call it the *limiting system* to the sequence $(g_1, g_2, \ldots)$ and (M, f). It is a topological factor of the product system $(M_1 \times M_2 \times \ldots, f_1 \times f_2 \times \ldots)$ and is again in $\mathcal{F}$. The limiting system construction can also be done in the invertible setup.

Proposition 4.1. Assume the starting system (M, f) has a shift embedded. Given groups G_i containing $\mathbb{Z}$ and a sequence of maps g_i which are invariant under the $\mathbb{Z}$-shift, then there exists a sequence ϵ_i such that if $||g_i - \mathrm{Id}|| < \epsilon_i$, then (M_∞, f_∞) has shifts of any dimension embedded.

Proof. Choose inductively $||g_i - \mathrm{Id}|| < \epsilon_i$ so small that $\phi(M_j, f_j)$ has at least a $(j+1)$-dimensional shift embedded. □

Fix the initial manifold M and the set of groups G_i. The limiting system (M_∞, f_∞) depends on $f \in C(M, M)$ as well as the maps $g_i \in C(M_{i-1}^{G_i}, M_{i-1}^{G_i})$.

Proposition 4.2. The map $(f, \{g_i\}) \mapsto f_\infty$ is a continuous map from $C(M, M) \times \oplus_i C(M_{i-1}^{G_i}, M_{i-1}^{G_i})$ to $C(M_\infty, M_\infty)$.

Proof. Map ϕ_i is continuous from $C(M_{i-1}, M_{i-1})$ to $C(M_i, M_i)$. Therefore, the product system continuously depends on f and g_i. The passage from the product system to (M_∞, f_∞) is a continuous factor map. □

5. Discussion

We took here the point of view that the coupled map lattice construction is a dynamical system on some space $\mathcal{F}$ of topological dynamical systems. (Other examples of dynamical systems on dynamical systems are the Feigenbaum renormalisation map on unimodal maps or cellular automata on subshifts.)

Starting with a trivial system, this leads to a hierarchy of in general nonlinear systems which define a limiting system of high complexity.

Assume G_i is constant and that ϕ is such that it makes sense on any space M^G. If we consider the map ϕ as a kind of renormalisation map on the class $\mathcal{F}$ of dynamical systems, it is natural to ask, what is the attractor in $\mathcal{F}$ of ϕ. If the map ϕ would have the property that the limiting system is independent of the starting system (M, f), then the limiting map (M_∞, f_∞) can be considered as the unique fixed point of ϕ.

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