

Ch. VI. Extremal length.

1. Let  $G$  be a region and  $\Gamma$  a set whose elements  $\gamma$  are rectifiable arcs in  $G$  or, more generally, sums of such arcs. If  $G$  is mapped conformally on  $G'$ ,  $\Gamma$  is transformed into a set  $\Gamma'$ . Our object is to find a number  $\lambda(\Gamma, G)$  with the invariance property  $\lambda(\Gamma, G) = \lambda(\Gamma', G')$ .

It is natural to focus the attention on the lengths of the curves  $\gamma$ . However, length is not conformally invariant. Therefore we use an arbitrary metric  $ds = \rho |dz|$  which is conformal to the Euclidean metric. Under a conformal mapping  $z \rightarrow z'$  it is transformed into  $\rho' |dz'|$  where  $\rho' = \rho \left| \frac{dz}{dz'} \right|$ . We assume that  $\rho$  is defined throughout  $G$ , that  $\rho \geq 0$ , and that  $\rho$  is lower semicontinuous:  $\rho(z_0) \leq \liminf_{z \rightarrow z_0} \rho(z)$ . Under these conditions every  $\gamma \in \Gamma$  has a  $\rho$ -length

$$L(\gamma, \rho) = \int_{\gamma} \rho |dz|$$

which may be infinite. Similarly, it is possible to define a  $\rho$ -area

$$A(G, \rho) = \iint_G \rho^2 dx dy .$$

For quantities that correspond to each other under the conformal mapping  $z \rightarrow z'$  it is clear that  $L(\gamma, \rho) = L(\gamma', \rho')$  and  $A(G, \rho) = A(G', \rho')$ .

2. In order to simplify we introduce

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho) .$$

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2. In order to simplify we introduce

$$l(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho)$$

This is to be compared with  $A(G, \rho)$ . Since  $L(\Gamma, \rho)$  is homogeneous of degree 1 and  $A(G, \rho)$  is homogeneous of degree two in  $\rho$  it is natural to form the ratio  $L(\Gamma, \rho)^2/A(G, \rho)$ . The set of all such ratios is a conformal invariant. But it is evident that the ratio can be made arbitrarily small. Therefore, the set is characterized by its maximum, and we are led to the following definition:

Definition. The extremal length of  $\Gamma$  in  $G$  is defined by

$$\lambda_G(\Gamma) = \sup_{\rho} \frac{L(\Gamma, \rho)^2}{A(G, \rho)}$$

where  $\rho$  is subject to the condition  $0 < A(G, \rho) < \infty$ .

In the definition it is possible to use different normalizations, obtained by multiplying  $\rho$  with suitable constants. For instance,  $\lambda_G(\Gamma)$  is equal to  $\sup L(\Gamma, \rho)^2$  subject to the condition  $0 < A(G, \rho) \leq 1$ . It is also equal to the reciprocal of  $\inf A(G, \rho)$  when  $L(\Gamma, \rho) \geq 1$ .

Another convenient normalization is expressed by the condition  $L(\Gamma, \rho) = A(G, \rho)$ . Due to the different degrees of homogeneity it is always fulfilled for a suitable multiple of a given  $\rho$ , except when  $L(\Gamma, \rho) = 0$  or  $\infty$ . With this normalization we have

$$\lambda_G(\Gamma) = \sup L(\Gamma, \rho) = \sup A(G, \rho).$$

With the right interpretation, the exceptional cases are included. For instance,  $\lambda_G(\Gamma) = 0$  means that  $L(\Gamma, \rho) = 0$  whenever  $A(G, \rho)$  is finite and positive; if we exclude infinite values, the only case of equality is thus  $L(\Gamma, \rho) = A(G, \rho) = 0$ .

2. The conformal invariance is an immediate consequence of the definition. We want to point out, in addition, that in a sense  $\lambda_G(\Gamma)$  depends only on

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Definition. The extremal length of  $\Gamma$  in  $G$  is defined by

$$\lambda_0(\Gamma) = \sup_{\rho} \frac{L(\Gamma, \rho)}{A(G, \rho)^{1/2}}$$

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In the definition it is possible to use different normalizations, obtained by multiplying  $\rho$  with suitable constants. For instance,  $\lambda_0(\Gamma)$  is equal to  $\sup L(\Gamma, \rho)$  subject to the condition  $0 < A(G, \rho) \leq 1$ . It is also equal to the reciprocal of  $\inf A(G, \rho)$  when  $L(\Gamma, \rho) \leq 1$ . Another convenient normalization is expressed by the condition  $L(\Gamma, \rho) = A(G, \rho)$ . Due to the different degrees of homogeneity it is always fulfilled for a suitable multiple of a given  $\rho$ , except when  $L(\Gamma, \rho) = 0$  or  $\infty$ . With this normalization we have

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$\Gamma$  and not on  $G$ . To see this, suppose that  $G \subset G'$ . Given  $\rho$  on  $G$  we define  $\rho' = \rho$  on  $G$  and  $\rho' = 0$  on  $G - G'$ . Then  $\rho'$  is lower semi-continuous, and  $L(\Gamma, \rho) = L(\Gamma, \rho')$ ,  $A(G, \rho) = A(G, \rho')$ . This proves that  $\lambda_{G'}(\Gamma) \geq \lambda_G(\Gamma)$ . For the opposite conclusion we need only start from a  $\rho'$  in  $G'$  and let  $\rho$  be its restriction to  $G$ . We see that  $\lambda_G(\rho)$  is the same for all open sets that contain the curves  $\Gamma$ .

3. To illustrate, we consider the more special notion of extremal distance. Let  $G$  be a region, and let  $E_1, E_2$  be two sets in the closure of  $G$ . They may be taken to be disjoint, since everything would otherwise be trivial. Now take  $\Gamma$  to be the set of connected arcs in  $G$  which join  $E_1$  to  $E_2$ . Then  $d_G(E_1, E_2) = \lambda(\Gamma)$  is called the extremal distance of  $E_1$  and  $E_2$  with respect to  $G$ . It depends essentially on  $G$  because the set  $\Gamma$  depends on  $G$ .

A typical example is the quadrilateral. It is a Jordan region with four distinguished boundary points. They divide the boundary into two pairs of opposite sides  $(\alpha, \alpha')$  and  $(\beta, \beta')$ . The quadrilateral is oriented by choosing one of these pairs as the base pair, for instance  $(\alpha, \alpha')$ . We consider the extremal distance between the sides  $\alpha$  and  $\alpha'$ ; this is also called the module of the oriented quadrilateral.

The quadrilateral  $Q$  can be mapped on a rectangle  $R$  by a conformal mapping  $w = u + iv = f(z)$ . We assume that the  $\alpha$ -sides of  $R$  lie on  $u = 0$ ,  $u = a$  and the  $\beta$ -sides on  $v = 0$ ,  $v = b$ . The first observation is that  $\rho = 1$  satisfies  $L(\Gamma, 1) = a$ ,  $A(R, 1) = ab$ . This gives  $d \geq a/b$ . Conversely, let  $\rho$  be arbitrary but normalized by  $L(\Gamma, \rho) = a$ , say. Then

$$\int_0^a (\rho(u, v) - 1) du \geq 0$$

and not on  $G$ . To see this, suppose that  $G \subset G'$ . Given  $\gamma$  on  $G$  we define  $\gamma' = \gamma$  on  $G$  and  $\gamma' = 0$  on  $G - G'$ . Then  $\gamma'$  is lower semi-continuous, and  $L(\gamma, \rho) = L(\gamma', \rho) = A(G, \rho) = A(G', \rho)$ . This proves that  $L(G, \rho) \leq L(G', \rho)$ . For the opposite conclusion we need only start from  $\gamma'$  in  $G'$  and let  $\gamma$  be its restriction to  $G$ . We see that  $L(G, \rho) \leq L(G', \rho)$  is the same for all open sets that contain the curves  $\gamma$ .

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$$\int_0^a \int_0^b \rho(u, v) (1 - i) du \leq 0$$

and hence

$$\iint_R (\rho - 1) du dv \geq 0 .$$

Together with

$$\iint_R (\rho - 1)^2 du dv \geq 0$$

it follows that  $A(R, \rho) \geq ab$ , and hence that  $d \leq a/b$ . We have proved that  $d$  is equal to the ratio of the sides in a rectangle conformally equivalent to  $Q$ . The result proves that the ratio is uniquely determined. If we interchange  $\alpha$  and  $\beta$  we get  $d^* = b/a$ . Observe that  $dd^* = 1$ .

4. We prove the important

Comparison principle. If every  $\gamma \in \Gamma$  contains a  $\gamma' \in \Gamma'$ , then  $\lambda(\Gamma) \geq \lambda(\Gamma')$ .

Both extremal lengths can be evaluated with respect to the same  $G$ . For any  $\rho$  in  $G$  it is clear that  $L(\Gamma, \rho) \geq L(\Gamma', \rho)$ . These minimum lengths are compared with the same  $A(G, \rho)$ , and the assertion follows.

When the comparison principle is applied to extremal distance we find:

The extremal distance  $d_G(E_1, E_2)$  decreases when  $G$ ,  $E_1$  and  $E_2$  increase.

It should be noted, however, that the comparison principle is more general, even when applied only to extremal distances.

5. The comparison principle is very similar to the majorization principle for harmonic measure. Extremal length has the additional advantage that we can

and hence

$$\int_R \psi - 1) \leq 0$$

Together with

$$\int_R (\psi - 1)^2 \leq 0$$

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The extremal distance  $d(E_1, E_2)$  decreases when  $G, E_1$  and  $E_2$  increase.

It should be noted, however, that the comparison principle is more general, even when applied only to extremal distances.

5. The comparison principle is very similar to the majorization principle for harmonic measure. Extremal length has the additional advantage that we can

formulate two composition laws as follows:

Theorem. Let  $G_1$  and  $G_2$  be disjoint open sets. Let  $\Gamma_1, \Gamma_2$  consist of curves in  $G_1, G_2$  respectively, and let  $\Gamma$  be a third family.

a) If every  $\gamma \in \Gamma$  contains a  $\gamma_1 \in \Gamma_1$  and a  $\gamma_2 \in \Gamma_2$ , then

$$\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2).$$

b) If every  $\gamma_1 \in \Gamma_1$  and every  $\gamma_2 \in \Gamma_2$  contains a  $\gamma \in \Gamma$ , then

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

If one of the extremal lengths degenerates, being either 0 or  $\infty$ , the statements are trivial consequences of the comparison principle. To prove a) in a non-degenerate case, choose  $\rho_1$  in  $G_1$  and  $\rho_2$  in  $G_2$ , and let them be normalized so that  $L(\Gamma_i, \rho_i) = A(G_i, \rho_i)$ ,  $i = 1, 2$ . Suppose that  $G \supset G_1 \cup G_2$ . We put  $\rho = \rho_1$  in  $G_1$ ,  $\rho = \rho_2$  in  $G_2$  and  $\rho = 0$  in  $G - G_1 - G_2$ . Then  $L(\Gamma, \rho) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$  and  $A(G, \rho) = A(G_1, \rho_1) + A(G_2, \rho_2) = L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$ . Hence  $\lambda(\Gamma) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$ , and a) follows.

As for b), let  $\rho$  be given in  $G$  and normalized by  $L(\Gamma, \rho) = 1$ . Then  $L(\Gamma_1, \rho) \geq 1$ ,  $L(\Gamma_2, \rho) \geq 1$ , and this implies

$$A(G, \rho) \geq A(G_1, \rho) + A(G_2, \rho) \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

On the other hand, the greatest lower bound of  $A(G, \rho)$  is  $1/\lambda(\Gamma)$ .



6. In the case of the rectangle we found that  $\rho_0 = 1$  is an extremal metric in the sense that  $L(\Gamma, \rho_0)^2/A(R, \rho_0)$  is equal to its maximum  $\lambda(\Gamma)$ . More generally, for the quadrilateral the extremal metric is  $\rho_0 = |f'(z)|$  where  $f$  is the mapping function on a rectangle. We note that the extremal metric is euclidean.

If  $\Gamma' \subset \Gamma$ , when can one assert that  $\lambda(\Gamma') = \lambda(\Gamma)$ ? We know by the comparison principle that  $\lambda' \geq \lambda$ . Now, if there is an extremal metric  $\rho_0$  for  $\Gamma'$ , and if it satisfies  $L(\Gamma, \rho_0) = L(\Gamma', \rho_0)$ , then it is evident that  $\lambda \geq \lambda'$ , so that  $\lambda = \lambda'$ .

For instance, let  $G$  be a doubly connected region. By conformal mapping it can be transformed into an annulus  $r_1 < |z| < r_2$ . We consider the extremal distance between the boundary circles. Cut the annulus along a radius. Then it becomes a quadrilateral with the module  $\log r_2/r_1 : 2\pi$ . The extremal metric is  $\rho_0 = 1/|z|$ , and all the shortest curves are in the smaller set of curves that do not cross the cut. Therefore the extremal distance in the annulus is the same as in the quadrilateral.

We can also consider the extremal length of the family of curves that separate the two contours. The same reasoning shows that it is  $2\pi : \log r_2/r_1$ .

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the two contours. The same reasoning shows that it is  $2\pi : \log r_2/r_1$ .

Ch.VII. Systematic study of the one parameter case.

1. We shall use extremal length to determine modules of configurations. There is a good formula for the number of conformal invariants which we shall regard as an empirical fact. Suppose we have a region with  $m$  contours,  $n_1$  distinguished interior points and  $n_2$  distinguished points on the boundary. Then the number of modules is

$$N = 3m - 6 + 2n_1 + n_2 + (s)$$

where  $s$  is usually zero. Actually,  $s$  is the number of parameters in the family of conformal self-mappings. Thus it is 3 for a disk, 1 for an annulus and for a disk with two boundary points.

The cases with  $N = 1$  are the most accessible with our method. We determine the integer solutions of  $3m + 2n_1 + n_2 = 7$ . They are

1.  $m = 1$     $n_1 = 2$     $n_2 = 0$
2.  $m = 1$     $n_1 = 1$     $n_2 = 2$
3.  $m = 1$     $n_1 = 0$     $n_2 = 4$
4.  $m = 2$     $n_1 = 0$     $n_2 = 1$

Case 1. Simply connected region with two inner points. We already have a conformal invariant, namely the Green's function  $g(z_1, z_2)$ . Hence every other must be a function of this one. Nevertheless, it is instructive to form other invariants and compare them with the Green's function.



Case 2. Simply connected region with one interior point and two boundary points. A conformal invariant is the harmonic measure at the interior point of the arc between the boundary points. Same remark as above.

Case 3. The quadrilateral. We have already considered the module. Another invariant is obtained by mapping on a disk or a halfplane and forming the cross-ratio of the boundary points.

Case 4. Annulus with one boundary point. The boundary point is of no interest since we can move it to any position by a rotation. Therefore this is just the case of the annulus, and we have already determined its module.

2. We take a closer look at case 1. Assume that the region is a disk and that  $z_1, z_2$  are on the real diameter. As a conformal module we introduce  $\lambda^* = \lambda(\Gamma^*)$  where  $\Gamma^*$  is the family of closed curves  $\gamma^*$  which separate  $z_1$  and  $z_2$  from the circumference. Let  $z_1, z_2$  be joined by a line segment  $s$ . We introduce the smaller family  $\Gamma_0^*$  which separates  $s$  from the circumference. Now we are dealing with an annulus, and we can compute the extremal length  $\lambda_0^*$ . There is an extremal metric  $\rho_0$  which is obviously symmetric with respect to the real axis.

We contend that  $\lambda^* = \lambda_0^*$ . To prove it we need only show that  $L(\Gamma^*, \rho_0) = L(\Gamma_0^*, \rho_0)$ . Given a  $\gamma^*$  we get a  $\gamma_0^*$  of equal length by reflecting part of  $\gamma^*$  over the real axis. This is all that is needed. We have made important use of the symmetry of the extremal metric.

We can look at the result a little differently. Let  $C$  be any continuum that contains  $z_1$  and  $z_2$ . Let  $d_C$  be the extremal distance from  $C$  to the circle. Then  $\max d_C$  is clearly a conformal invariant. We have, on passing to the conjugate extremal length,

Case 2. Simply connected region with one interior point and two boundary points. A conformal invariant is the harmonic measure at the interior point of the arc between the boundary points. Same remark as above.

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We contend that  $\lambda^* = \lambda_0^*$ . To prove it we need only show that  $L(T^*, \rho_0^*) = L(T_0^*, \rho_0^*)$ . Given a  $\gamma^*$  we get a  $\gamma_0^*$  of equal length by reflecting part of  $\gamma^*$  over the real axis. This is all that is needed. We have made important use of the symmetry of the extremal metric.

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$$d_C^* \geq \lambda^* = \lambda_0^* = d_S^*$$

and thus  $d_C \leq d_S$ .

Quite similarly, we can determine the extremal length  $\lambda$  of the curves that separate  $z_1, z_2$  and have their end points on the circumference. This time we draw the complementary segments on the real diameter. We can then consider the disk as a quadrilateral, and the extremal distance between the two semicircles is obviously  $2d_S = 2/\lambda_0^*$ . We call this distance  $\lambda_0$ , and exactly the same reasoning as above shows that  $\lambda = \lambda_0$ . Note that  $\lambda \lambda^* = 2$ .

Let us determine the functional dependence between  $\lambda$  and  $g(z_1, z_2)$ . For this purpose we introduce a standard transcendental function  $\Delta(\rho)$ , defined for  $\rho > 0$ , which is equal to the extremal distance of the segments  $(0, 1)$  and  $(\rho, \infty)$  with respect to the upper half-plane. It is obvious that  $\Delta$  increases with  $\rho$  from 0 to  $\infty$ . We assume for simplicity that  $z_1 = 0$ ,  $z_2 = z_0 > 0$ . The function  $\gamma = \left(\frac{1-z}{1+z}\right)^2$  takes  $(-1, 0, z_0, 1)$  into  $(\infty, 1, \gamma_0, 0)$ . Thus  $\lambda^*/2 = \Delta(\log \gamma_0^{-1}) = \Delta(4d(z_1, z_2))$  where  $d$  is the non-euclidean distance.

3. In case 2 we can take the interior point as the origin, and we suppose that the boundary points are end points of an arc of length  $\alpha$  with center at 1. Here we take  $\Gamma$  to be the family of arcs which begin and end on the given arc and separate the origin from the complementary arc. We draw the negative radius and let  $\lambda_0$  be the extremal <sup>distance</sup> length between the radius and the arc. One finds by the same kind of reasoning that  $\lambda = 4\lambda_0$ . One can again compute  $\lambda$  explicitly. This time  $z = e^{i\alpha/2}$  corresponds to  $\gamma = -\tan^2 \alpha/4$ , and we are led to the relation

$$\lambda = \frac{2}{\Delta\left(2 \log \sec \frac{\alpha}{2}\right)}$$



Ch. VIII. Schlicht functions in rings.

(These lectures were taken from an ancient manuscript which is appended.

Watch for slight changes in the notation.)