Ch. IV

Boundary Correspondence.

A. The $\mathfrak{M}$-condition.

We have shown that a q.c. mapping of a disk on itself induces a topological mapping of the circumference. How regular is this mapping? Can it be characterized by some simple condition? The surprising thing is that it can.

Things become slightly simpler if we map the upper halfplane on itself and assume that ∞ corresponds to ∞. The boundary correspondence is then given by a continuous increasing real valued function $h(x)$ such that $h(-\infty) = -\infty$ and $h(+\infty) = +\infty$. What conditions must it satisfy?

We suppose first that there exist a K-q.c. mapping φ of the upper halfplane on itself with boundary values $h(x)$. It can immediately be extended by reflection to a K-q.c. mapping of the whole plane, and we are therefore in a position to apply the results of the preceding chapter. Namely, let $e_1 < e_3 < e_2$ be real points which are mapped on e_1', e_3', e_2'. If

$$\rho = \frac{e_3 - e_1}{e_2 - e_1} \quad , \quad \rho' = \frac{e_3' - e_1'}{e_2' - e_1'}$$

and τ, τ' are the corresponding values on the imaginary axis we have

$$K^{-1} \operatorname{Im} \tau \leqq \operatorname{Im} \tau' \leqq K \operatorname{Im} \tau .$$

We shall use only the very simplest case where e_1, e_3, e_2 are equidistant points $x-t, x, x+t$ and consequently $\rho = \frac{1}{2}$ which corresponds to $\tau = i$. In this case we have thus

$$K^{-1} \leqq \operatorname{Im} \tau(\rho') \leqq K .$$

Equivalently, this means that

$$\rho(iK^{-1}) \leqq \rho' \leqq \rho(iK)$$

or

$$(1) \quad 1-\rho(iK) \leqq \rho' \leqq \rho(iK).$$

Actually, what we prefer are bounds for

$$\frac{e_2' - e_3'}{e_3' - e_1'} = \frac{1-\rho'}{\rho'}$$

and ~~the~~ from (1) we get

$$\frac{1-\rho(iK)}{\rho(iK)} \leqq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqq \frac{\rho(iK)}{1-\rho(iK)}$$

Even better, let us recall that $\rho(\tau+1) = \frac{1}{\rho(\tau)}$. For this reason the lower bound can be written as $\rho(1+iK)-1$ and by our product formula (Ch III) we find

$$\rho(1+iK)-1 = 16\, e^{-\pi K} \prod_{1}^{\infty} \left(\frac{1+e^{-2n\pi K}}{1-e^{-(2n-1)\pi K}} \right)^8$$

We have thus proved

$$(2) \qquad M(K)^{-1} \leqq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqq M(K)$$

where

$$(3) \qquad M(K) = \frac{1}{16}\, e^{\pi K} \prod_{1}^{\infty} \left(\frac{1-e^{-(2n-1)\pi K}}{1+e^{-2n\pi K}} \right)^8 .$$

We call (2) an M-condition. Obviously (3) gives the best possible value. The upper bound

$$M(K) \leqq \frac{1}{16}\, e^{\pi K} .$$

<u>Theorem</u>. The boundary values of a K-q.c. mapping satisfy the $M(K)$-condition.

It is important to study the consequences of an M-condition

$$(4) \quad M^{-1} \leqq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqq M$$

even quite apart from its importance for q.c. mappings. Let $H(M)$ denote the family of all such h. Observe that it is invariant under linear transformations $S: x \to ax+b$ both of the dependent and the independent variables. In other words, if $h \in H(M)$ then $S_1 \circ h \circ S_2 \in H(M)$. We shall let $H_0(M)$ denote the subset of functions h that are normalized by $h(0) = 0$, $h(1) = 1$.

For $h \in H_0(M)$ we have immediately

$$(5) \quad \frac{1}{M+1} \leqq h\left(\tfrac{1}{2}\right) \leqq \frac{M}{M+1}$$

and by induction this generalizes to

$$(6) \qquad \frac{1}{(M+1)^n} \leqq h\left(\frac{1}{2^n}\right) \leqq \left(\frac{M}{M+1}\right)^n .$$

This is true for negative n as well. In other words, we have for instance

$$h(2^n) \leqq \left(1 + \frac{1}{M}\right)^n$$

which shows that the $h \in H_0(M)$ are uniformly bounded on any compact intervals (apply to $h(x)$ and $1 - h(1-x)$)

They are also equicontinuous. Indeed, for any fixed a the function

$$\frac{h(a+x) - h(a)}{h(a+1) - h(a)}$$

is normalized. Hence $0 \leqq x \leqq \frac{1}{2^n}$ gives

$$h(a+x) - h(a) \leqq (h(a+1) - h(a))\left(\frac{M}{M+1}\right)^n$$

which proves the equicontinuity on compact sets. As a consequence

<u>Lemma 1.</u> The space $H_0(M)$ is compact (under uniform convergence on compact sets).

Indeed, the limit function of a sequence must satisfy (4), and this immediately makes it strictly increasing.

Actually, the compactness characterizes the $H_0(M)$'.

<u>Lemma 2.</u> Let H_0 be a set of normalized homeomorphisms h which is compact and stable under composition with linear mappings. Then $H_0 \subset H_0(M)$ for some M.

<u>Proof.</u> Set $\alpha = \inf h(-1)$, $\beta = \sup h(-1)$ for $h \in H_0$. There exists a sequence such that $h_n(-1) \to \alpha$, and a subsequence which converges to a homeomorphism. Therefore $\alpha > -\infty$, and the same reasoning gives $\beta < 0$.

For any $h \in H_0$ the mapping

$$k(x) = \frac{h(y+tx) - h(y)}{h(y+t) - h(y)}, \quad t > 0$$

is in H_0. Hence

$$\alpha \leqq \frac{h(y-t) - h(y)}{h(y+t) - h(y)} \leqq \beta$$

or

$$-\frac{1}{\alpha} \leqq \frac{h(y+t) - h(y)}{h(y) - h(y-t)} \leqq -\frac{1}{\beta}$$

which is an M-condition.

We shall also need the following more specific information:

Lemma 3. If $h \in H_0(M)$ then

$$\frac{1}{M+1} \leqq \int_0^1 h(x)\,dx \leqq \frac{M}{M+1}.$$

Proof. Let us set $F(x) = \sup h(x)$, $h \in H_0(M)$. This is a curious function that seems very difficult to determine explicitly. However, some estimates are easy to come by.

We have already proved that $F(\frac{1}{2}) \leqq \frac{M}{M+1}$. Because

$$\frac{h(tx)}{h(t)} \in H_0(M)$$

we obtain, for $x = \frac{1}{2}$,

$$\frac{h(\frac{t}{2})}{h(t)} \leqq F(\tfrac{1}{2})$$

and hence

$$(7) \qquad F(\tfrac{t}{2}) \leqq F(\tfrac{1}{2})\,F(t) \quad \text{for} \quad t>0$$

Similarly

$$\frac{h((1-t)x+t)-h(t)}{1-h(t)} \in H_0(M)$$

gives

$$\frac{h(\frac{1+t}{2})-h(t)}{1-h(t)} \leqq F(\tfrac{1}{2}).$$

For $t < 1$ this gives

$$h\left(\frac{1+t}{2}\right) \leqq F\left(\frac{1}{2}\right) + \left(1 - F\left(\frac{1}{2}\right)\right) h(t)$$

and

$$(8) \qquad F\left(\frac{1+t}{2}\right) \leqq F\left(\frac{1}{2}\right) + \left(1 - F\left(\frac{1}{2}\right)\right) F(t).$$

Adding (7) and (8)

$$(9) \qquad F\left(\frac{t}{2}\right) + F\left(\frac{1+t}{2}\right) \leqq F\left(\frac{1}{2}\right) + F(t).$$

Now

$$\int_0^1 F(t)\,dt = \frac{1}{2}\int_0^2 F\left(\frac{t}{2}\right) dt$$

$$= \frac{1}{2}\int_0^1 \left(F\left(\frac{t}{2}\right) + F\left(\frac{1}{2} + \frac{t}{2}\right)\right) dt \leqq \frac{1}{2} F\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^1 F(t)\,dt$$

Hence

$$\int_0^1 F(t)\,dt \leqq F\left(\frac{1}{2}\right)$$

and the assertion follows.

The opposite inequality follows on applying the result to $1 - h(1-t)$.

<u>Remark.</u> From

$$\frac{1}{M+1} \leqq h\left(\tfrac{1}{2}\right) \leqq \frac{M}{M+1}$$

the weaker inequalities

$$\frac{1}{2(M+1)} \leqq \int_0^1 h\,dt \leqq \frac{2M+1}{2(M+1)}$$

are immediate, and since they serve the same purpose Lemma 3 is a luxury.

B. The sufficiency

We shall prove the converse:

Theorem Every mapping h which satisfies an M-condition is extendable to a K-q.c. mapping for a K that depends only on M.

The proof is by an explicit construction. We shall indeed set $\varphi(x,y) = u(x,y) + i v(x,y)$ where

$$(1)\qquad u(x,y) = \frac{1}{2y} \int_{-y}^{y} h(x+t)\,dt$$

$$v(x,y) = \frac{1}{2y} \int_{0}^{y} \left(h(x+t) - h(x-t)\right) dt .$$

It is clear that $v(x,y) \geq 0$ and tends to 0 for $y \to 0$. Moreover, $u(x,0) = h(x)$, as desired.

The formulas may be rewritten

(1)′

$$u = \frac{1}{2y} \int_{x-y}^{x+y} h(t)\,dt$$

$$v = \frac{1}{2y} \left(\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt \right)$$

and in this form it is evident that the partial derivatives exist and are

$$u_x = \frac{1}{2y} \left(h(x+y) - h(x-y) \right)$$

$$u_y = -\frac{1}{2y^2} \int_{x-y}^{x+y} h\,dt + \frac{1}{2y} \left(h(x+y) + h(x-y) \right)$$

$$v_x = \frac{1}{2y} \left(h(x+y) + 2h(x) + h(x-y) \right)$$

$$v_y = -\frac{1}{2y^2} \left(\int_{x}^{x+y} h\,dt - \int_{x-y}^{x} h\,dt \right) + \frac{1}{2y} \left(h(x+y) - h(x-y) \right).$$

The following simplification is possible: if we replace $h(t)$ by $h_1(t) = h(at+b)$, $a > 0$, the M-condition remains in force and $\varphi(z)$ is replaced by $\varphi_1(z) = \varphi(az+b)$. Thus $\varphi_1(i) = \varphi(ai+b)$, and since $ai+b$ is arbitrary we need only study the dilatation at the point i. Also, we are still free to choose $h(0) = 0$, $h(1) = 1$.

The derivatives now become

$$u_x = \frac{1}{2}\left(1 - h(-1)\right)$$

$$u_y = - \frac{1}{2}\int_{-1}^{1} h\,dt + \frac{1}{2}\left(1 + h(-1)\right)$$

$$v_x = \frac{1}{2}\left(1 + h(-1)\right)$$

$$v_y = -\frac{1}{2}\left(\int_0^1 h\,dt - \int_{-1}^{0} h\,dt\right) + \frac{1}{2}\left(1 - h(-1)\right).$$

The (small) dilatation is given by

$$k = \left| \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)} \right|$$

To simplify we are going to set

$$\xi = 1 - \int_0^1 h\,dt$$

$$\beta = -h(-1)$$

$$\eta\beta = -h(-1) + \int_{-1}^{0} h\,dt \quad (>0).$$

Thus

$$u_x = \tfrac{1}{2}(1+\beta)$$

$$u_y = \tfrac{1}{2}(\xi - \eta\beta)$$

$$v_x = \tfrac{1}{2}(1-\beta)$$

$$v_y = \tfrac{1}{2}(\xi + \eta\beta)$$

giving

$$d \quad k = \left| \frac{\big((1-\xi) + \beta(1-\eta)\big) + i\big((1+\xi) - \beta(1+\eta)\big)}{\big((1+\xi) + \beta(1+\eta)\big) + i\big((1-\xi) - \beta(1-\eta)\big)} \right|$$

change k to d!

$$k^2 = \frac{1+\xi^2+s^2(1+\eta^2)-2s(\xi+\eta)}{1+\xi^2+s^2(1+\eta^2)+2s(\xi+\eta)}$$

$$\frac{1+k^2}{1-k^2} = \frac{1}{2}\left[\frac{1}{s}\,\frac{1+\xi^2}{\xi+\eta} + s\,\frac{1+\eta^2}{\xi+\eta}\right]$$

We have proved the estimates

$$M^{-1} \leqq s \leqq M$$

$$\frac{1}{M+1} \leqq \xi \leqq \frac{M}{M+1}$$

$$\frac{1}{M+1} \leqq \eta \leqq \frac{M}{M+1}$$

(the last follows by symmetry).

This gives, for instance

$$\frac{1+k^2}{1-k^2} < M(M+1)$$

$$D \cancel{K} < 2M(M+1).$$

As soon as we have this estimate we know that the Jacobian is positive, from which it follows that the mapping φ is locally one to one.

It must further be shown that $\varphi(z) \to \infty$ for $z \to \infty$. By (1') we have

$$u = \frac{1}{2y}\left(\int_{x-y}^{x} h\,dt + \int_{x}^{x+y} h\,dt\right)$$

$$v = \frac{1}{2y}\left(\int_{x}^{x+y} h\,dt - \int_{x-y}^{x} h\,dt\right)$$

$$u^2+v^2 = \frac{1}{2y^2}\left[\left(\int_{x}^{x+y} h\,dt\right)^2 + \left(\int_{x-y}^{x} h\,dt\right)^2\right]$$

If $x \geqq 0$ it is $> \frac{1}{2y^2}\left(\int_0^y h\,dt\right)^2$

If $x \leqq 0$ — $> \frac{1}{2y^2}\left(\int_{-y}^{0} h\,dt\right)^2$

and both tend to ∞ for $y \to \infty$. When y is bounded it is equally clear that $u^2+v^2 \to \infty$.

Now $\zeta = \varphi(z)$ defines the upper halfplane as a smooth unlimited covering of itself. By the monodromy theorem it is a homeomorphism.

1. Remark: Beurling and A. prove $D < M^2$. To do so they had to introduce an extra parameter in the definition of φ.

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2. Remark. It may be asked how regular a function $h(x)$ is that satisfies an M-condition. For a long time it was believed that the boundary correspondence would always be absolutely continuous. But this is not so, for it is possible to construct functions h that satisfy the M-condition without being absolutely continuous.

C. Quasi-isometry.

For conformal mappings of a halfplane onto itself non euclidean distances are invariant. For q.c.-mappings, are they quasi-invariant. Of course not. If non euclidean distances are multiplied by a bounded factor (in both directions) we shall say that the mapping is quasi-isometric.

Theorem. The mapping φ constructed in B is quasi-isometric. Indeed, it satisfies

(2) $$A^{-1} d[z_1, z_2] \leqq d[\varphi(z_1), \varphi(z_2)] \leqq A\, d[z_1, z_2]$$

with a constant A that depends only on M.

It is sufficient to prove (2) infinitesimally, that is,

(9) $$A^{-1}\frac{|dz|}{y} \le \frac{|d\varphi|}{v} \le A\frac{|dz|}{y}$$

Again, affine mappings of the z-plane are of no importance, and it is therefore sufficient to consider the point $(0,1)$. We have

$$v(i) = \frac{1}{2}\left(\int_0^1 h\,dt - \int_{-1}^0 h\,dt\right)$$

and the estimates

$$\frac{1}{2M} \le v(i) \le \frac{M+1}{2}$$

are immediate (using the Lemma).

This is to be combined with

$$|d\varphi| \le |\varphi_z|\,|dz|$$

$$|\varphi_z|^2 = 2\left[(1+\xi^2)+s^2(1+\eta^2)+2s(\xi+\eta)\right]$$

$$y(M+1)^2$$

and

$$|d\varphi| \geq |\varphi_{\bar z}||d\bar z| \geq \frac{1}{D}|\varphi_z||dz|.$$

One finds that (2) holds with

$$A = 4M(M+1).$$

(Details left to reader).

D. Quasiconformal reflection.

Consider now a K-q.c. mapping φ of the whole plane onto itself. The real axis is mapped on a simple curve L that goes to ∞ in both direction. Is it possible to characterize L through geometric properties.*)

First some general remarks. Suppose that L divides the plane into Ω and Ω^* corresponding to the upper halfplane H and the lower halfplane L^*. Let j denote the reflection $z \to \bar{z}$ that interchanges H and H^*. Then $\varphi \circ j \circ \varphi^{-1}$ is a sense reversing K^2-q.c. mapping which interchanges Ω, Ω^* and keeps L pointwise fixed. We say that L admits a K^2-q.c. reflexion.

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*) For instance, we know already that L has zero area.

Conversely, suppose that L admits a K-q.c. reflexion ω. Let f be a conformal mapping from H to Ω. Define

$$(1)\quad \begin{cases} F = f & \text{in } H \\ F = \omega \circ f \circ j & \text{in } H^*. \end{cases}$$

It is clear that F is K-q.c. So we see that L admits a reflection if and only if it is the image of a line under a q.c. mapping of the whole plane. Moreover, we are free to choose this mapping so that it is conformal in one of the half-planes. We say that the conformal mapping f admits a K-q.c. extension to the whole plane.

We can also consider the conformal mapping f^* from H^* to Ω^*. The mapping $j \circ f^{*-1} \circ w \circ f$ is a quasiconformal mapping of H on itself. Its restriction to the x-axis is $h(x) = f^{*-1} \circ f$, and we know that it must satisfy an M-condition. Observe that L determines h uniquely except for linear transformation ($h(x)$ can be replaced by $A\,h(ax+b) + B$).

On the other hand, suppose that h is given and satisfies an M-condition. We know that there exists a q.c. mapping with these boundary values: we make it a sensereversing mapping ι from H to H^*. We cannot yet prove it, but there exists a mapping φ of the whole plane upon itself which is conformal in H and such that $\varphi \circ \iota \circ j$ is conformal in H^* (this condition determines

μ_φ in the whole plane, and one of our main theorems will be to ~~prove~~ that we can determine φ when μ_φ is given). This φ maps the real axis on a line L which in turn determines h.

How unique is L? Suppose L_1 and L_2 admit q.c. reflections ω_1 and ω_2 and denote the corresponding conformal mappings by f_1, f_1^*, f_2, f_2^*. Assume that they determine the same $h = f_1^{*-1} \circ f_1 = f_2^{*-1} \circ f_2$. The mapping

$$g = \begin{cases} f_2 \circ f_1^{-1} & \text{in } \Omega_1 \\ f_2^* \circ f_1^{*-1} & \text{in } \Omega_1^* \end{cases}$$

is conformal in $\Omega_1 \cup \Omega_1^*$ and continuous on L_1. Is it conformal in the whole plane? To prove that this is so we show that g is quasiconformal, for we know that a q.c. mapping which is conformal a.e. is conformal.

Introduce F_1 and F_2 as in (1). Form

$$G = F_2^{-1} \circ f_2^* \circ f_1^{*-1} \circ F_1 \quad {}^{1)}$$

in H^*. It reduces to identity on the real axis, and we set $G = z$ in H. Then G is q.c. Hence $F_2 \circ G \circ F_1^{-1}$ is quasiconformal. It reduces to $f_2^* \circ f_1^{*-1}$ in Ω_1^* and to $f_2 \circ f_1^{-1}$ in Ω_1, that is, to g.

We conclude that g is conformal. Hence f_2 differs from f_1 only by a linear transformation, and L is essentially unique.

There are two main problems:

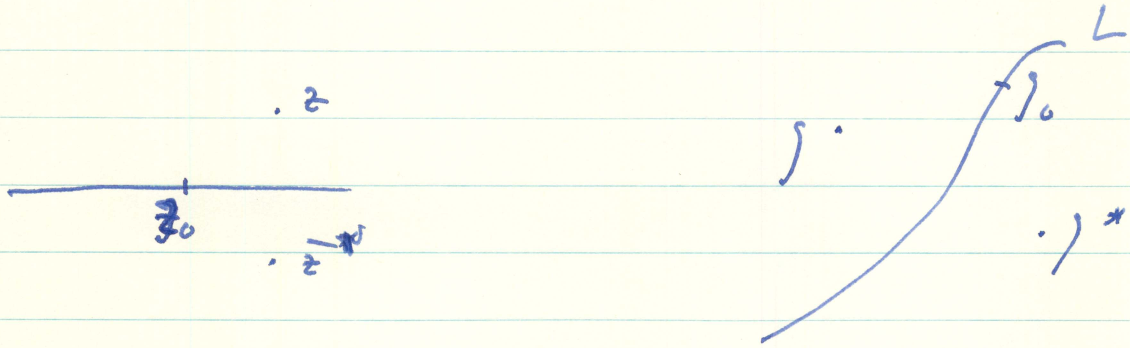
<u>Problem 1.</u> To characterize L by geometric properties.

<u>Problem 2.</u> To characterize f (and f^*)

1) $G = j \circ f_2^{-1} \circ \omega_2 \circ f_2^* \circ f_1^{*-1} \circ \omega_1 \circ f_1 \circ j$

We shall solve Probl. 1. I don't know how to solve Probl. 2. The characterization should be in analytic properties of the invariant $\frac{f''}{f'}$.

We begin by showing that a K-q.c. reflection retains many characteristics of an ordinary reflection.



We shall set $\zeta^* = \omega(\zeta)$ and suppose that $\zeta = \varphi(z)$, $\zeta^* = \varphi(\bar z)$, $\zeta_0 = \varphi(x_0)$ when $\varphi(x_0)$ is real.

Numerical functions of K alone, not only the same, will be denoted by $C(K)$.

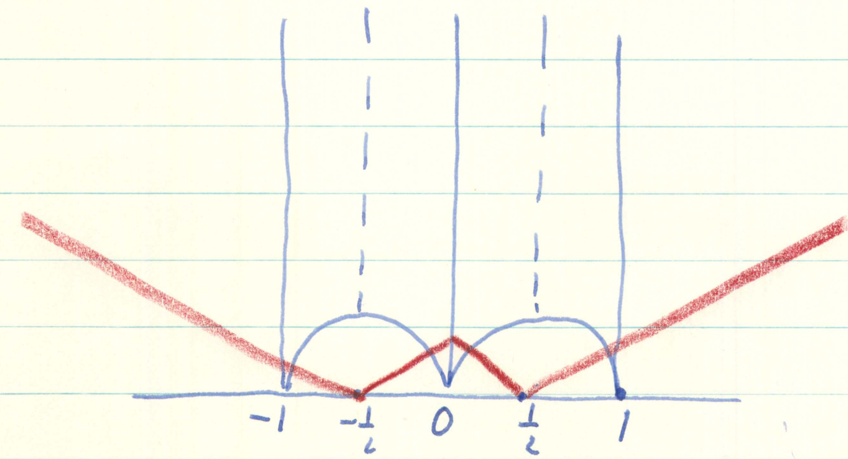
<u>Lemma 1.</u> $$C(K)^{-1} \le \left|\frac{\zeta^* - \zeta_0}{\zeta - \zeta_0}\right| \le C(K)$$

<u>Proof.</u> We note that

$$\rho = \frac{z_0 - z}{z_0 - \bar{z}}$$

satisfies $|\rho| = 1$. For any such ρ there is a corresponding τ situated on the lines $\operatorname{Re} \tau = \pm \frac{1}{2}$, $\operatorname{Im} \tau \geq \frac{1}{2}$.

We conclude that there is a τ' corresponding to $\rho' = (\rho_0 - \rho)/(\rho_0 - \rho^*)$ that is within n.e. distance $\log K$ of these lines. This means that τ' lies in a W-shaped region indicated in our figure



We note that the points ± 1 where $\rho = \infty$ are shielded from the W-region. Since $\rho \to 1$ as $\operatorname{Im} \tau \to \infty$ and $\rho \to -1$ at the points $\pm \frac{1}{2}$, provided that they are approached within an angle. Therefore ρ is bounded by a constant $C(K)$, and this proves the lemma.

Remark: It is not necessary to investigate the behavior of $\rho(\tau)$ at $\tau = \pm \frac{1}{2}$, for the true region to which τ' is restricted does not reach these points.

Let $\delta(\zeta)$ be the shortest euclidean distance from ζ to L.

Lemma 2. $$C(K)^{-1} \leq \frac{\delta(\zeta^*)}{\delta(\zeta)} \leq C(K)$$

Proof trivial.

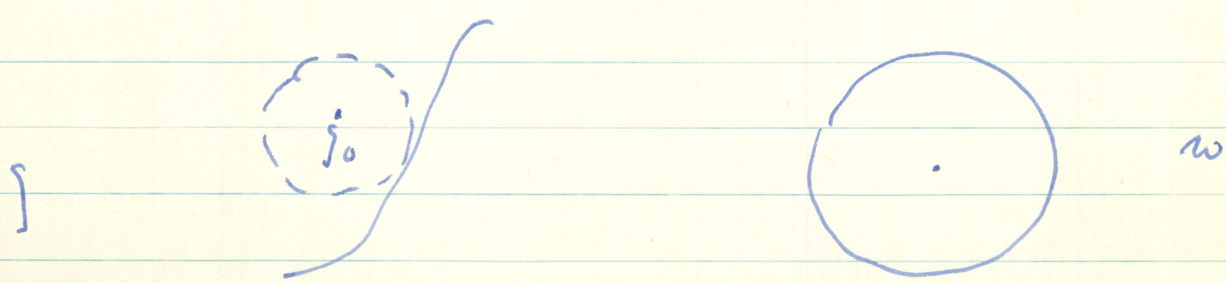
The regions Ω and Ω^* carry their own noneuclidean metric defined by ~~the same~~

$$\lambda |d\zeta| = \frac{|dz|}{y}$$

for $\zeta = f(z)$. The reflection ω induces a K-q.c. mapping of H on H^*, and we know that it can be changed to a $C(K)$ quasi-isometric mapping. It follows that we can replace ω by a reflection ω' such that (at points $\zeta^* = \omega'(\zeta)$)

$$C(K)^{-1} \lambda |d\zeta| \leq \lambda^* |d\zeta^*| \leq C(K) \lambda |d\zeta|$$

But it is elementary to estimate $\lambda(\zeta)$ in terms of $\delta(\zeta)$.



For this purpose map Ω conformally on $|w|<1$ with $w(\zeta_0)=0$. Schwarz lemma gives

$$|w'(\zeta_0)| \leqq \frac{1}{\delta(\zeta_0)} .$$

But the non euclidean line element at the origin is $2\,|dw|$. So

$$\lambda(\zeta_0) = 2\,|w'(\zeta_0)| \leqq \frac{2}{\delta(\zeta_0)} .$$

In the other direction Koebe's distortion theorem gives

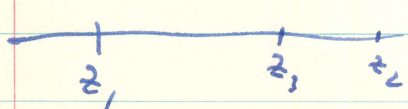
$$\delta(\zeta_0) \geqq \frac{1}{4}\,\frac{1}{|w'(\zeta_0)|} ,$$

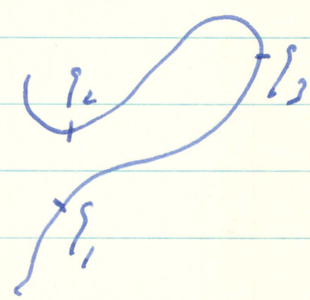
$$\lambda(\zeta_0) \geqq \frac{1}{2\,\delta(\zeta_0)} .$$

Combining the results with Lemma 2 we conclude

Lemma 3. If there exists a K-q.c. reflection across L, then there is also a $C(K)$-q.c. reflection which is differentiable and changes euclidean lengths at most by a factor $C(K)$.

This is a surprising result, for a priori one would expect the stretching to ~~vary~~ satisfy only a Hölder condition.

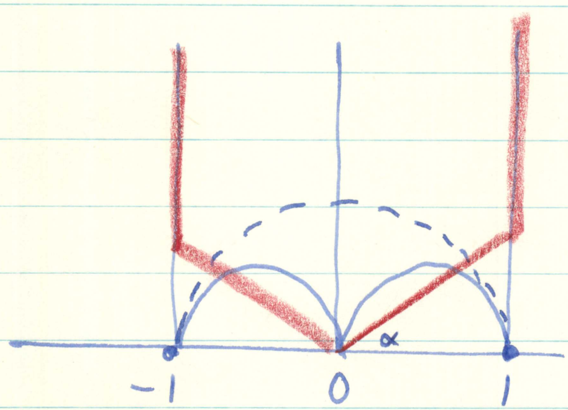




Now consider three points on L such that ζ_3 lies between ζ_1, ζ_2. Now $\rho = \dfrac{z_1 - z_3}{z_1 - z_2}$ is between 0 and 1 which means that τ lies on the imaginary axis. Therefore τ^* lies in an angle

$$2\arctan K^{-1} \leqq \arg \tau^* \leqq \pi - 2\arctan K^{-1}.$$

It can also be chosen so that $|\mathrm{Re}\,\tau^*| \leqq 1$ so τ^* is restricted to the following region



Again it is obvious that $|\rho|$ has a maximum $C(K)$ and we have proved:

<u>Theorem.</u> If $\zeta_1, \zeta_3, \zeta_2$ are any three points on L such that ζ_3 separates ζ_1, ζ_2 then

$$\left|\frac{\zeta_3 - \zeta_1}{\zeta_1 - \zeta_2}\right| \leqq C(K).$$

It is more symmetric to write

$$\left| \zeta_3 - \frac{\zeta_1 + \zeta_2}{2} \right| \leqq C(K) \, |\zeta_1 - \zeta_2|$$

and in this form the best value of $C(K)$ can be computed. It corresponds to the point $e^{i\alpha}$ and can be computed explicitly.

E. The reverse inequality.

We shall prove that the condition in the last theorem is not only necessary, but also sufficient. In other words:

Theorem. A necessary and sufficient condition for L to admit a q.c. reflection is the existence of a constant C such that

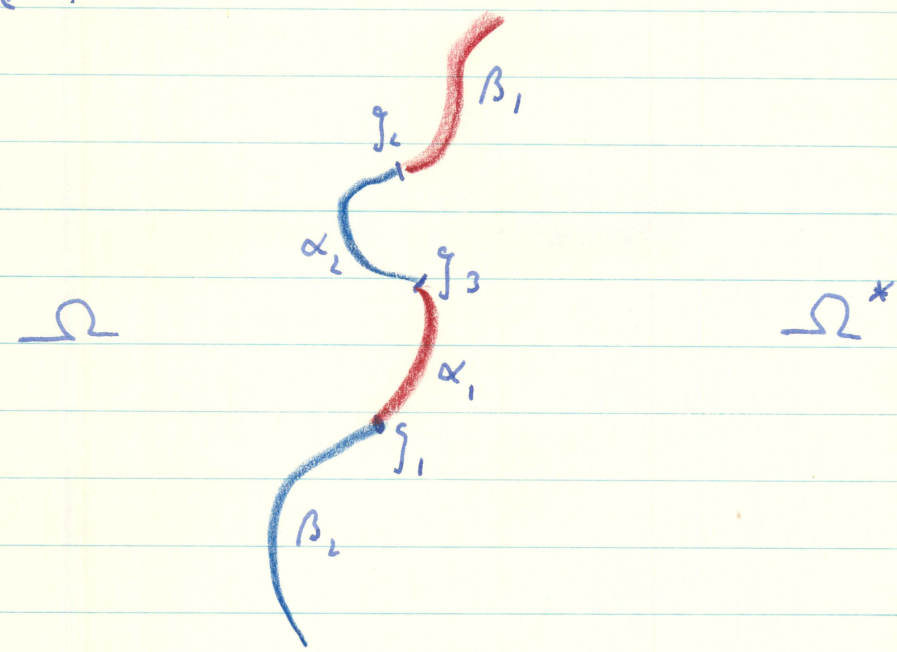
$$(\ ) \qquad \frac{|\zeta_3 - \zeta_1|}{|\zeta_2 - \zeta_1|} \leqq C$$

for any three points on L such that ζ_3 is between ζ_1 and ζ_2.

More precisely, if K is given C depends only on K, and if C is given there is a K-q.c. reflection where K depends only on C.

In my paper „Quasiconformal reflections" in Acta Mathematica I proved this by elementary ~~use~~ estimates of extremal length. It seems more appropriate to make systematic use of our mapping $\rho \to \tau$.

Introduce notations by this figure:



Let λ_k be the extremal distance from α_k to β_k in Ω, and λ_k^* the corresponding distance in Ω^*. Then $\lambda_1 \lambda_2 = 1$, $\lambda_1^* \lambda_2^* = 1$. Through the conformal mapping of Ω, let $\zeta_1, \zeta_3, \zeta_2$ correspond to $x-t$, x, $x+t$. This means that $\lambda_1 = \lambda_2 = 1$. Through the conformal mapping of Ω^* they correspond to $h(x-t)$, $h(x)$, $h(x+t)$. If we can show that λ_1^* is bounded it follows at once that h satisfies an M-condition, and hence that a q.c. reflection exists.

Let Γ_1 be the class of simple closed curve that separate ξ_2 and ξ_3 from ξ_1. Each curve in Γ_1 contains an arc that joins α_1 and β_1. Hence

$$\lambda(\Gamma_1) \geqq \lambda_1$$

$$\lambda(\Gamma_2) \geqq \lambda_2$$

On the other hand, we know that

$$\lambda(\Gamma_1) = \frac{2}{\operatorname{Im} \tau}$$

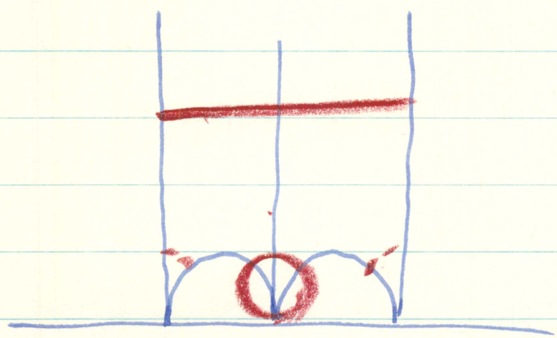
when τ corresponds to the ratio $\rho = \dfrac{\xi_3 - \xi_1}{\xi_2 - \xi_1}$ and lies in the fundamental region. Γ_2 corresponds similarly to $1-\rho$, and hence to

$-\frac{1}{\tau}$. For $\lambda_1 = \lambda_2 = 1$ we have thus

$$\operatorname{Im} \tau \leqq 2$$

$$\operatorname{Im}\left(-\frac{1}{\tau}\right) \leqq 2 .$$

The figure shows the region to which τ is confined:



But we have still another condition, namely $\rho \leqq C$. This cuts off the cusps at ± 1, and τ is confined to a compact set.

By consideration of Ω^* we have similarly, for instance,

$$\frac{2}{\operatorname{Im} \tau} \geqq \lambda_1^* .$$

But $\operatorname{Im} \tau$ has a positive minimum, hence

λ_1^* has a finite upper bound, and the theorem is proved.