Unit 20: Line integral theorem

Lecture

20.1. When a vector field is integrated along a curve, we get a **line integral**. In the special case where $\vec{F}$ is a gradient field, we get a theorem which generalizes the fundamental theorem of calculus.

**Definition:** If $\vec{F}$ is a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $C : t \mapsto \vec{r}(t)$ is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

is called the **line integral** of $\vec{F}$ along the curve $C$.

20.2. We use also the short-hand notation $\int_C \vec{F} \cdot d\vec{r}$. In physics, if $\vec{F}(x, y, z)$ is a **force field**, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$ is **work**. In electrodynamics, if $\vec{F}(x, y, z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$ is the **electrostatic potential**.

20.3. Let $C : t \mapsto \vec{r}(t) = [\cos(t), \sin(t)]$ be a circle parameterized by $t \in [0, 2\pi]$ and let $\vec{F}(x, y) = [-y, x]$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.

**Solution:** We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) \, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt = 2\pi$
20.4. Let \( \vec{r}(t) \) be a curve given in polar coordinates as \( \vec{r}(t) = [r(t), \phi(t)] = [\cos(t), t] \) defined on the interval \( 0 \leq t \leq \pi \). Let \( \vec{F} \) be the vector field \( \vec{F}(x, y) = [-xy, 0] \). Calculate the line integral \( \int_C \vec{F} \cdot d\vec{r} \). Solution: In Cartesian coordinates, the curve is \( \vec{r}(t) = [\cos^2(t), \cos(t)\sin(t)] \). The velocity vector is then \( \vec{r}'(t) = [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)] = (x(t), y(t)) \). The line integral is
\[
\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^\pi [\cos^2(t)\sin(t), 0] \cdot [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)] \, dt
\]
\[
= -2 \int_0^\pi \sin^2(t)\cos^4(t) \, dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8.
\]

20.5. The first generalization of the fundamental theorem of calculus to higher dimensions is the fundamental theorem of line integrals.

**Theorem:** Fundamental theorem of line integrals: If \( \vec{F} = \nabla f \), then
\[
\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = f(\vec{r}(b)) - f(\vec{r}(a)).
\]

20.6. In other words, the line integral is the potential difference between the end points \( \vec{r}(b) \) and \( \vec{r}(a) \), if \( \vec{F} \) is a gradient field.

**Examples**

20.7. Let \( f(x, y, z) \) be the temperature distribution in a room and let \( \vec{r}(t) \) the path of a fly in the room, then \( f(\vec{r}(t)) \) is the temperature, the fly experiences at the point \( \vec{r}(t) \) at time \( t \). The change of temperature for the fly is \( \frac{df}{dt} f(\vec{r}(t)) \). The line-integral of the temperature gradient \( \nabla f \) along the path of the fly coincides with the temperature difference between the end point and initial point.

20.8. Here are some special cases: If \( \vec{r}(t) \) is parallel to the level curve of \( f \), then \( \frac{df}{dt} f(\vec{r}(t)) = 0 \) because \( \vec{r}'(t) \) is orthogonal to \( \nabla f(\vec{r}(t)) \). If \( \vec{r}(t) \) is orthogonal to the level curve, then \( |\frac{df}{dt} f(\vec{r}(t))| = |\nabla f| |\vec{r}'(t)| \) because \( \vec{r}'(t) \) is parallel to \( \nabla f(\vec{r}(t)) \).

20.9. The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:
\[
\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_a^b \frac{df}{dt}(\vec{r}(t)) \, dt = f(\vec{r}(b)) - f(\vec{r}(a)).
\]

**Theorem:** For a gradient field, the line-integral along any closed curve is zero.

20.10. When is a vector field a gradient field? \( \vec{F}(x, y) = \nabla f(x, y) \) implies \( P_y(x, y) = Q_x(x, y) \). If this does not hold at some point, \( \vec{F} = [P, Q] \) is no gradient field. This is called the Clairaut test. We will see later that the condition \( \text{curl}(\vec{F}) = Q_x - P_y = 0 \) and \( \vec{F} \) being defined everywhere implies that the field is a gradient field.
20.11. Let $\vec{F}(x, y) = [2xy^2 + 3x^2, 2yx^2]$. Find a potential $f$ of $\vec{F} = [P, Q]$.

**Solution:** The potential function $f(x, y)$ satisfies $f_x(x, y) = 2xy^2 + 3x^2$ and $f_y(x, y) = 2yx^2$. Integrating the second equation gives $f(x, y) = x^2y^2 + h(x)$. Partial differentiation with respect to $x$ gives $f_x(x, y) = 2xy^2 + 3x^2$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x, y) = x^2y^2 + x^3$. Find $g, h$ from $f(x, y) = \int_0^x P(x, y) \, dx + h(y)$ and $f_y(x, y) = g(x, y)$.

20.12. Let $\vec{F}(x, y) = [P, Q] = \left[\frac{-xy}{x^2+y^2}, \frac{x}{x^2+y^2}\right]$. It appears to be a gradient field because $f(x, y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1 + y^2/x^2) = P, f_y = (1/x)/(1 + y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F} \, d\gamma$, where $\gamma$ is the unit circle is

$$\int_0^{2\pi} \left[\frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)}\right] \cdot \left[\sin(t), \cos(t)\right] \, dt$$

which is $\int_0^{2\pi} 1 \, dt = 2\pi$. What is wrong?

**Solution:** note that the potential $f$ as well as the vector-field $\vec{F}$ are not differentiable everywhere. The curl of $\vec{F}$ is zero except at $(0, 0)$, where it is not defined.

20.13. A device which implements a non gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up and to analyze why they don’t work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water. You find plenty of other futile attempts on youtube.
Problem 20.1: What is the work done by moving in the force field \( \vec{F}(x, y) = [3x^2 + 1, 8y^7] \) along the parabola \( y = x^2 \) from \((-1, 1)\) to \((1, 1)\)? In part a) compute it directly. Then, in part b), use the theorem.

Problem 20.2: Let \( C \) be the space curve \( \vec{r}(t) = [\cos(t), \sin(\sin(t)), 5t] \) for \( t \in [0, \pi] \) and let \( \vec{F}(x, y, z) = [y, x, 15] \). Find the value of the line integral \( \int_C \vec{F} \cdot d\vec{r} \). You might want to use a theorem.

Problem 20.3: Let \( \vec{F} \) be the vector field \( \vec{F}(x, y) = [-y, x]/2 \). Compute the line integral of \( F \) along an ellipse \( \vec{r}(t) = [a \cos(t), b \sin(t)] \) with width \( 2a \) and height \( 2b \). The result should depend on \( a \) and \( b \).

Problem 20.4: It is hot and you refresh yourself in a little pool in your garden. Its rim has the shape \( x^{40} + y^{40} = 1 \) oriented counter clockwise. There is a hose filling in fresh water to the tub so that there is a velocity field \( \vec{F}(x, y) = [2x + 5y, 10y^4 + 5x] \) inside. Calculate the line integral \( \int_C \vec{F} \cdot d\vec{r} \), the energy you gain from the fluid force when dislocating from \((1, 0)\) to \((0, 1)\) along the rim. Remember you are in a pool and do not want to work hard. There is an easy way to get the answer.

Problem 20.5: Find a closed curve \( C : \vec{r}(t) \) for which the vector field
\[
\vec{F}(x, y) = [P(x, y), Q(x, y)] = [xy, x^2]
\]
satisfies \( \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \neq 0 \).

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