Some remarks on signs in functional equations

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To Robert Rankin

Let k be a number field, and let M be a pure motive of weight n over k. Assume that there is a non-degenerate pairing $M \times M \to \mathbf{Q}(-n)$. For example, M could arise from the cohomology in degree n of a projective, smooth, geometrically connected variety X of dimension n over k, and the pairing from the cup product. In this case, the pairing will be symplectic (alternating) if n is odd, and orthogonal (symmetric) if n is even.

Then the conjectural functional equation for the complete L-function of M (including the local factors at infinity) takes the form (cf. [De, (1.2.3)])

$$L(M, s) = \epsilon(M, s)L(M, n + 1 - s). \tag{0.1}$$

Here $\epsilon(M, s) = \epsilon(M) \cdot f(M)^{(n+1)/2-s}$, with $f(M) \geq 1$ an integer, and $\epsilon(M) = \pm 1$ the sign of our title.

In the cases where $L(M, s)$ has an analytic interpretation, and its functional equation is known, the sign $\epsilon(M)$ is of considerable importance. We often get interesting formulae by computing $\epsilon(M)$ in distinct ways.

1. A famous example, illustrating the last point, is the calculation of the sign of the quadratic Gauss sum (cf. [Da, Ch 2]).

Let $k = \mathbf{Q}$, and assume M is orthogonal, of weight 0 and rank 1. Then M corresponds to a quadratic character of the Galois group, or equivalently, to a quadratic étale extension K. Let D be the discriminant of K.

Dirichlet reinterpreted the theorem of quadratic reciprocity to show that the factorization of a prime p in K depended only the class of p (mod D). Hence

$$L(M, s) = L(\chi, s) \tag{1.1}$$

where χ is a quadratic character of $(\mathbf{Z}/D\mathbf{Z})^*$, and $L(\chi, s)$ is the (complete) Dirichlet L-function.

Let $f = \chi(-1) \cdot D = |D|$, which is equal to the conductor $f(M)$, and define the quadratic Gauss sum:

$$g(\chi) = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi(a) e^{2\pi i a/f}. \tag{1.2}$$

Let $\sqrt{D}$ be the unique square-root of D in $\mathbf{C}$, which is positive when $D > 0$, and has positive imaginary part when $D < 0$. Then Dirichlet's functional equation gives the formula [Da, Ch 9]:

$$\epsilon(M) = g(\chi)/\sqrt{D}. \tag{1.3}$$

In particular, since $\epsilon(M)^2 = 1$, this yields the algebraic identity of Gauss:

$$g(\chi)^2 = D = \chi(-1) \cdot f. \tag{1.4}$$

On the other hand, $L(M, s)$ is the quotient of the (complete) zeta function of K by the (complete) zeta function of $\mathbf{Q}$. Since Hecke's functional equation for any zeta function has sign $= 1$, we have

$$\epsilon(M) = +1 \tag{1.5}$$

$$g(\chi) = \sqrt{D}. \tag{1.6}$$

2. Another important result involving signs is the proof by Serre of Hecke's theorem on the class of the absolute different (cf. [W, pg. 291]).

The above argument shows that for any orthogonal motive M of rank 1 and weight 0 over k, $\epsilon(M) = +1$. Indeed, M corresponds to a quadratic, étale extension K, and $L(M, s)$ is the quotient of the zeta function of K by the zeta function of k.

Now assume further that K is everywhere unramified over k. By class field theory, M gives rise to a quadratic character χ of the ideal class group $C(k)$ of k, and $L(M,s)$ is equal to the Hecke L-series $L(\chi, s)$.

Hecke showed that the sign in the functional equation of $L(\chi, s)$ is $\chi(\mathcal{D})$, for any ideal class character χ, where $\mathcal{D}$ is the absolute different of k (cf. [T, pg. 94]). Hence, for all *quadratic* χ, we have

$$\chi(\mathcal{D}) = +1. \tag{2.1}$$

This implies that the class $[\mathcal{D}]$ of $\mathcal{D}$ is a *square* in $C(k)$, i.e. $[\mathcal{D}]$ lies in the subgroup $2C(k)$.

3. What can we say about the class of $\mathcal{D}$ in the strict class group $C^+(k)$—the quotient of the group of ideals by the subgroup of principal ideals with a totally positive generator? The following result was obtained in [G-M]; the proof given here follows ideas of Armitage and Fröhlich [A-F].

Proposition 3.1. *Let K/k be a quadratic extension which is unramified at all finite primes, and let χ be the corresponding quadratic character of $C(k)^+$. Let T be the set of real places of k which become complex in K.*

Then the integer $\#T$ is even, and $\chi(\mathcal{D}) = (-1)^{\#T/2}$. The class $[\mathcal{D}]$ of $\mathcal{D}$ is a square in $C(k)^+$ if and only if for every unramified, quadratic extension K of k, the integer $\#T$ is divisible by 4.

Proof. The sign in the functional equation of $L(\chi, s) = L(M, s)$ is given by ([T, pg. 94]):

$$\epsilon(M) = (-1)^{\#T/2} \cdot \chi(\mathcal{D}).$$

Since $L(M,s)$ is the quotient of two zeta functions, $\epsilon(M) = +1$, and $\chi(\mathcal{D}) = (-1)^{\#T/2}$ as claimed. The class $[\mathcal{D}]$ lies in $2C(k)^+$ if and only if $\chi(\mathcal{D}) = 1$ for all quadratic characters χ of $C(k)^+$.

4. We end with some remarks on signs of functional equations for abelian varieties. Let A be an abelian variety of dimension g over k, and let $M_A = H_1(A)$, which is a motive of weight -1 and rank $2g$. The Weil pairing gives a non-degenerate, alternating bilinear form

$$M_A \times M_A \to \mathbf{Q}(1). \tag{4.1}$$

Let M_ρ be an orthogonal motive of weight 0 and rank 2 over k. Then M_ρ corresponds to a complex Galois representation of the form $\mathrm{Ind}_K(\rho)$. Here K is the quadratic étale extension corresponding to the rank 1 orthogonal motive $\det(M_\rho)$, and ρ is a character of $\mathrm{Gal}(\overline{k}/K)$ with trivial transfer to $\mathrm{Gal}(\overline{k}/k)$. We view $\chi = \det(M_\rho)$ as a generalized ideal class character of k, and let T denote the set of real places of k which become complex in K.

Put $M = M_A \otimes M_\rho$, which is symplectic of weight -1 and rank $4g$. Local considerations (cf. [T, pg. 115]) lead one to predict that when the conductor ideal f_A of A is prime to the conductor of M_ρ,

$$\epsilon(M) = \chi(f_A) \cdot (-1)^{g \cdot \#T}. \tag{4.2}$$

In fact, Rankin's method gives a proof of the functional equation of $L(M, s)$, with the above sign, when k is totally real and A appears as a new factor of the Jacobian of a Shimura curve of level N over k. Then $f_A = (N)^g$, and $L(M, s)$ is the product of g Rankin L-series, each with sign $\chi(N) \cdot (-1)^{\#T}$. The simplicity of this sign was the starting point for my work with Zagier on the conjecture of Birch and Swinnerton-Dyer.

5. Bibliography

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