Rigid local systems on $\mathbb{G}_m$ with finite monodromy

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Katz has constructed some rigid local systems of rank 7 on $\mathbb{G}_m$ in finite characteristics, whose monodromy is a finite subgroup G of the complex Lie group M of type G_2 [K, Thm 9.1] . Among the five finite subgroups G of M which appear in his work, four occur in their natural characteristic:

$SL_2(8)$ in char 2

$PU_3(3)$ in char 3

$PGL_2(7)$ in char 7

$PSL_2(13)$ in char 13.

In this paper, we give another construction of these four local systems on $\mathbb{G}_m$, and find similar local systems with finite monodromy in other complex Lie groups. Some examples of finite groups which occur in exceptional complex Lie groups M, analogous to the ones above are

$$G = PSL_2(27) \quad \text{in char } 3 \quad M = F_4$$

$$G = PSU_3(8) \quad \text{in char } 2 \quad M = E_7$$

$$G = PGL_2(31) \quad \text{in char } 31 \quad M = E_8$$

$$G = PSL_2(61) \quad \text{in char } 61 \quad M = E_8$$

There are interesting families in the classical groups. For example, when $q = p^f > 2$, we have:

$$G = PU_3(q) \ \text{ in char } p \qquad M = Sp_{2n} \quad 2n = q(q-1).$$

Our method of construction is as follows. Let k be the finite field with q elements. We first consider the action of a finite group $G = G(q)$ of the type

$$G = PGL_2(q) = A_1(q)$$

$$G = PU_3(q_0) = {}^2A_2(q) \quad q = q_0^2$$

$$G = Sz(q) = {}^2B_2(q) \qquad q = 2q_0^2$$

$$G = R(q) = {}^2G_2(q) \qquad q = 3q_0^2$$

on the corresponding Deligne-Lusztig curve Y over k. (We note that the algebraic group $PU_3(q_0)$ is defined over q_0, but is naturally a subgroup of $A_2(q) = PGL_3(q)$ with $q = q_0^2$.) This extends to an action of G on the complete curve $X = Y \cup S$, where S denotes the finite set of points in the completion. Using results of Lusztig [L], we show the quotient map $X \to X/G$ is ramified over precisely 2 points of the base, which has genus 0. Identifying X/G with $\mathbb{P}^1$, and the two points of ramification with $\{0, \infty\}$, we obtain a surjective homomorphism

$$f : \pi_1(\mathbb{G}_m/k) \to G,$$

where $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$.

We then study embeddings

$$g : G \to M$$

where M is a simple, complex Lie group. Let Ad be the adjoint representation of M on its Lie algebra. The composite homomorphism

$$\pi_1(\mathbb{G}_m/k) \xrightarrow[f]{} G \xrightarrow[g]{} M \xrightarrow[Ad]{} GL(\mathrm{Lie}(M))$$

gives a complex respresentation of the Galois group of the global field $k(\mathbb{G}_m) = k(t)$, which is unramified outside $t = 0, \infty$. We say the homomorphism g is *rigid* if the Artin L-function $L(\mathrm{Ad}, s)$ is equal to the constant function 1.

The embedding g is rigid precisely when the ℓ-adic sheaf on $\mathbb{P}^1$ corresponding to the middle extension of the adjoint representation has vanishing cohomology. This is equivalent to the statement that the principal M-bundle on $\mathbb{G}_m$ has no deformations, preserving the singularities at $t = 0$ and $t = \infty$.

Using results of Serre [S] and Greiss-Ryba [G-Ry], we construct rigid embeddings for $G = PGL_2(q)$ and $G = PU_3(q_0)$, as well as for their normal subgroups

$$G_0 = PSL_2(q) \quad q \equiv 1 \pmod 2.$$

$$G_0 = PSU_3(q_0) \quad q_0 \equiv 2 \pmod 3.$$

These give rise to the rigid local systems on $\mathbb{G}_m$ with finite monodromy, described above.

As a bonus, we obtain simple wild parameters (cf. [G-Re, §5.6]) for the local field $k((1/t))$, by restricting our local systems to the decomposition group at $t = \infty$. These parameters are representations of the local Galois group into M, which have no inertial invariants on the adjoint representation and have Swan conductor equal to the rank of M.

1. Deligne-Lusztig curves

Let $d = 2, 3, 4, 6$. For each d, and certain prime powers q, we consider the Deligne-Lusztig curve $Y = Y(w)$ over the field k with q elements. the restrictions on q are

$$\begin{array}{rll} d = & 2 & q = \text{arbitrary} \\ & 3 & q = q_0^2 \\ & 4 & q = 2q_0^2 \\ & 6 & q = 3q_0^2 \end{array}$$

The curve Y is affine and non-singular, and admits an action of the finite adjoint group $G = G(q)$ over k. In ATLAS notation, G is given by

$$\begin{array}{rll} d = & 2 & PGL_2(q) = A_1(q) \\ & 3 & PU_3(q_0) = {}^2A_2(q) \\ & 4 & Sz(q) = {}^2B_2(q) \\ & 6 & R(q) = {}^2G_2(q) \end{array}$$

The order of G is given by the formula

$$\#G = q^{d/2}(q^{d/2} + 1)(q - 1).$$

Let $\overline{k}$ be an algebraic closure of k and $F : G(\overline{k}) \to G(\overline{k})$ the (geometric) Frobenius map with fixed points G. When $d = 2, 3$ let $\sigma = F$; when $d = 4, 6$ let σ be a square-root of F. By definition, Y is the variety of all Borel subgrops of $G(\overline{k})$ such that B and $\sigma(B)$ are in relative position w, where w is the Coxeter class in the Weyl group relative to σ. The integer d is the order of a Coxeter element. (In [L] the affine curve Y is denoted X_f, and the integer d is denoted h_0.) Since G acts on the function field $k(Y)$, the action on Y extends to an action on the complete non-singular model $X = \overline{Y} = Y \cup S$, where S denotes the finite set of points in the completion.

There are two maximal subgroups of G that will play an important role in the study of the G action on X. The first is the Borel subgroup $B = B(q)$, of order $q^{d/2}(q-1)$. This fixes a point in a 2-transitive permutation representation, which we will see occurs on the points S of the completion. The p-Sylow subgroup U of B is normal, and the quotient B/U is cyclic. The second maximal subgroup is the normalizer $N(T)$ of a Coxeter torus $T = T(q)$. This is the semi-direct product of the cyclic group T, of order

$$\begin{array}{rll} d = & 2 & q+1 \\ & 3 & q - q_0 + 1 \\ & 4 & q - 2q_0 + 1 \\ & 6 & q - 3q_0 + 1 \end{array}$$

dividing $(q^{d/2}+1)$, with a Coxeter element of order d in G: $N(T) = T.d$.

The adjoint group $G = G(q)$ has a normal subgroup G_0, which is the image of k-rational points of the simply-connected cover. This has index > 1 in the following cases

$$\begin{array}{lll} d = 2 & PGL_2(q) \rhd_2 PSL_2(q) & q \equiv 1 \pmod 2 \\ d = 3 & PU_3(q_0) \rhd_3 PSU_3(q_0) & q_0 \equiv 2 \pmod 3 \end{array}$$

Both B and T have a non-trivial image in the quotient group G/G_0 in these cases, so the kernel defines maximal subgroups B_0 and $N(T_0) = T_0.d$ in G_0.

The action of G on the affine curve Y is completely described in [L]. We summarize these results in Theorem 1 below. For $m \geqslant 1$, let k_m be the unique extension of k of degree m in $\overline{k}$.

Theorem 1 (Lusztig)

a) *The set $Y(k_m)$ is empty, for $1 \leqslant m < d$.*

b) *G acts transitively on $Y(k_d)$ with stabilizer T.*

c) *G acts freely on $Y(\overline{k}) - Y(k_d)$, and acts simply-transivitely on $Y(k_{d+1})$.*

d) *The linear action of G on the compactly supported ℓ-adic cohomology groups $H^i_c(Y/\overline{k}, \mathbb{Q}_\ell)$ commutes with the action of F, and the distinct eigenspaces for F give distinct unipotent representations W of G. A unipotent representation occurs in $H^i_c(Y)$ if and only if $\mathrm{Tr}(t|W) = (-1)^i$, for all regular semi-simple elements t in T. If such a representation occurs, its multiplicity in $H^i_c(Y)$ is equal to one.*

One can make part d) more explicit by enumerating the eigenvalues α of F on $H^i_c(Y)$, and the unipotent representation W_α that appears in the α-eigenspace. They are

$$\begin{array}{lll} \alpha = q & \text{on } H^2_c(Y) & W = \text{trivial representation} \\ \alpha = 1 & \text{on } H^1_c(Y) & W = \text{Steinberg representation} \\ & & \dim W = q^{d/2} \end{array}$$

There are $(d-2)$ further eigenvalues on $H^1_c(Y)$, which correspond to unipotent cuspidal representation

$$
\begin{array}{ll}
d=3 & \\
\alpha = -q_0 & \dim W = q_0(q_0-1) \\[2ex]
d=4 & \\
\alpha_\pm = -q_0(1\pm i) & \dim W_\pm = q_0(q-1) \\[2ex]
d=6 & \\
\alpha_\pm = -q_0\left(\frac{3\pm\sqrt{-3}}{2}\right) & \dim W_\pm = q_0(q^2-1) \\[2ex]
\beta_\pm = -q_0(\pm\sqrt{-3}) & \dim W_\pm = q_0(q-1)(q+3q_0+1)/2
\end{array}
$$

The two unipotent cuspidal representations W of the Ree groups with

$$\dim W = q_0(q-1)(q-3q_0+1)/2$$

do **not** appear in the cohomology of Y, as they restrict to the regular representation of T.

From Lusztig's fundamental results, we deduce the following facts on the complete curve $X = \overline{Y} = Y \cup S$.

Corollary

a) *G acts transitively on $X(k) = S(k) = S(\overline{k})$, with stabilizer B.*

b) *The covering $X \to X/G = Z$ is ramified over two rational points of the base, which has genus 0.*

c) *We may fix an isomorphism $Z \to \mathbb{P}^1$ over k so that the resulting cover $\pi : X \to \mathbb{P}^1$ satisfies*

$$\pi(X(k)) = \infty$$

$$\pi(X(k_d)) - X(k)) = 0$$

$$\pi(X(k_{d+1}) - X(k)) = 1$$

d) *The decomposition group D, inertia group I, and wild inertia group P at the points $\{\infty, 0, 1\}$ are*

$D_\infty = I_\infty = B \rhd P_\infty = U \quad I_\infty/P_\infty = (q-1)$

$D_0 = N(T) \rhd I_0 = T \quad P_0 = 1 \quad D_0/I_0 = (d)$

$D_1 = (d+1) \quad I_1 = 1.$

Proof. Let W_0 be the representation of G on the functions on S with values in $\mathbb{Q}_\ell$ and with sum equal to 0. Then W_0 appears in $H^1_c(Y)$, and the eigenvalues of F on W_0 are roots of unity. The only such eigenvalue is $\alpha = 1$, so $S(k) = S(\overline{k})$. This eigenspace affords the Steinberg representation of G, which is the functions on G/B with sum equal to 0. Hence G acts transitively on $S(k) = X(k)$ with stabilizer B.

Since G fixes no non-zero vector in $H^1(X) = \oplus_{i=1}^{d-2} W_i$ the quotient curve $Z = X/G$ has genus 0. The decomposition groups at $\{\infty, 0, 1\}$ and their inertia subgroups follows from the structure of the G-orbits on $X(\overline{k})$, and their fields of rationality. We note that the element of order $(d+1)$ generating D_1 is a torsion class of Kac type (2; 1, 1) in $G(q)$, just as the element of d normalizing T is of Coxeter type (1; 1, 1).

Let $K = k(\mathbb{P}^1) = k(t)$, and let $K' = k(X)$ be the Galois extension of K corresponding to the covering $X \to \mathbb{P}^1$. Let V be a complex representation of $G = \mathrm{Gal}(K'/K)$, and for each place v of K, let F_v be an geometric Frobenius element in the quotient G_v/I_v. We define the Artin L-function of V by an Euler product over the places v of K:

$$L(V,x) = \prod_v \det(1 - F_v x^{degv}|V^{I_v})^{-1}.$$

This is a rational function of x, and is a polynomial in x when $V^G = 0$. We define the Artin conductor $f_v(V)$ and the Swan conductor $sw_v(V)$ as in [S2, Ch VI]. Then

$$f_0(V) = \dim(V/V^T)$$

$$f_\infty(V) = \dim(V/V^B) + sw_\infty(V).$$

Since the degree of $L(V,x)$ is given by [W, Appendix V]

$$\deg L(V,x) = f_0(V) + f_\infty(V) - 2\,\dim V$$

we find the formula

$$sw_\infty(V) = \deg L(V,x) + \dim V^B + \dim V^T.$$

Lusztig's calculation of the cohomology of Y gives us a determination of the Artin L-functions of the irreducible representations W of G, as we have the factorization

$$\prod_{\substack{\text{irred} \\ W}} L(W,x)^{\dim W} = \zeta_{K'}(x) = \frac{\det(1 - Fx|H^1(X))}{(1-x)(1-qx)}.$$

Corollary. *If W is an irreducible representation of G, then $L(W,x)$ has degree = -2, 0, or 1. More precisely*

a) *if W is the trivial representation*

$$L(W,x) = \frac{1}{(1-x)(1-qx)}$$

b) *if W is a unipotent cuspidal representation, which occurs as the α-eigenspace for F in $H^1(X)$, then*

$$L(W,x) = (1-\alpha x)$$

c) *for all other irreducible representations W, $L(W,x) = 1$.*

Corollary. a) *If W is the trivial representation of G, or the quadratic character of $PGL_2(q)$ when $q \equiv 1 \pmod 2$, or one of the cubic characters of $PU_3(q_0)$ when $q_0 \equiv 2$ (mod) then $sw_\infty(W) = 0$.*

b) *If W is an irreducible representation of $PGL_2(q)$ and* $\dim W > 1$, *then $sw_\infty(W) = 1$.*

c) *If W is the unipotent cuspidal representation of $PU_3(q_0)$, of dimension $q_0^2 - q_0$, then $sw_\infty(W) = 1$.*

Proof. a) These are precisely the irreducible representations which are tamely ramified. For all other irreducible representations, $sw_\infty(W) \geqslant 1$.

b) For these representations, $L(W,x) = 1$ has degree 0, so $sw_\infty(W) = \dim W^B + \dim W^T$. For the Steinberg representation $W = St$ of dimension q, we have $\dim W^B = 1$ and $\dim W^T = 0$. For all other irreducibles (of dimensions $q-1$, q, $q+1$) we have $\dim W^B = 0$ and $\dim W^T = 1$.

c) In this case, $L(W,x) = (1 + q_0 x)$ has degree 1. We find $\dim W^B =$

dim $W^T = 0$, as W is cuspidal and the restriction of W to T is the regular representation minus the trivial character. Hence $sw_\infty(W) = 1$.

The Swan conductors of all other irreducible representations W of G grow with the size of q. For example, the Steinberg representation St of $G(q)$ has

$$\begin{aligned} sw_\infty(St) &= \dim St^B + \dim St^T \\ &= 1 + \dim St^T \\ &= (q^{d/2}+1)/\#T \\ &= \begin{cases} 1 & d=2 \\ q_0+1 & d=3 \\ q+2q_0+1 & d=4 \\ q^2+3qq_0+2q+3q_0+1 & d=6 \end{cases} \end{aligned}$$

The two unipotent cuspidal representations W of the Ree group ${}^2G_2(q)$ which do **not** appear in the cohomology of X have

$$sw_\infty(W) = \dim W^T = q_0\left(\frac{q-1}{2}\right).$$

We now make some observations about the subfields of $K' = k(X)$ fixed by certain subgroups of the Galois group G. In the cases where we have a normal subgroup $G \rhd G_0$:

$$PGL_2(q) \underset{2}{\rhd} PSL_2(q) \qquad q \equiv 1 \pmod 2.$$

$$PU_3(q_0) \underset{3}{\triangleright} PSU_3(q_0) \qquad q_0 \equiv 2 \pmod 3.$$

The fixed field K_0 is tamely ramified, of degree $d = 2$ or 3, over $K = k(t)$. Since it is also split at the place $t = 1$, we find

$$K_0 = k(\sqrt[d]{t}).$$

Since the fixed field K_0 of G_0 also has genus 0, we obtain d homomorphisms up to conjugacy

$$f_0 : \pi_1(\mathbb{G}_m/k) \to G_0.$$

These are all conjugate under the other action of G. The different classes of f_0 correspond to fixing a d^{th} root of t in K_0, or equivalently, to fixing a place above the place $t = 1$ in K.

Now consider the subfield M of K' fixed by the subgroup B of G:

$$\begin{array}{ccc} & & K' = k(X) \\ & B\diagup & \\ M & & \\ & \diagdown & \\ & & K = k(t) \end{array}$$

This has degree $(q^{d/2} + 1)$ over K. Since $H^1(X)^B = 0$, it also has genus 0. The zeta function of M factors

$$\zeta_M(x) = \zeta_K(x) L(St, x)$$

and the discriminant divisor $D(M/K)$ of M/K is given by the conductor of the Steinberg representation:

$$D(M/K) = f_\infty(St).(\infty) + f_0(St)(0).$$

We have

$$
\begin{aligned}
f_\infty(St) &= \dim(St/St^B) + sw_\infty(St) \\
&= (q^{d/2} - 1) + sw_\infty(St) \\
&= \begin{cases} q & d = 2 \\ q_0^3 + q_0 & d = 3 \\ q^2 + q + 2q_0 & d = 4 \\ q^3 + q^2 + 3qq_0 + 2q + 3q_0 & d = 6 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
f_0(St) &= \dim(St/St^T) \\
&= \begin{cases} q & d = 2 \\ q_0^3 - q_0 & d = 3 \\ q^2 - q - 2q_0 & d = 4 \\ q^3 - q^2 - 3qq_0 - 2q - 3q_0 & d = 6 \end{cases}
\end{aligned}
$$

The prime P_∞ factors in M as

$$P_\infty = R \cdot (R')^{q^{d/2}} \qquad \text{with} \quad \deg R = \deg R' = 1$$

The prime P_0 factors in M as

$$P_0 = \prod_{i=1}^{n} (R_i)^{\#T} \quad \deg R_i = 1$$

where $n = (q^{d/2} + 1)/\#T$. All other primes are unramified.

In the case where $G = PGL_2(q)$, the curve X is the projective line over $k = \mathbb{F}_q$ and the action is by fractional linear transformations. Here, specific polynomials of degree $(q+1)$ over $K = k(t)$ are known, with root field M and splitting field $K' = k(X)$. Can one find such equations, of degree $(q^{d/2}+1)$ over K, in general?

2. Rigid local systems

In the previous section, we defined a surjective homomorphism

$$f : \pi_1(\mathbb{G}_m/k) \to G$$

where k is a finite field with q elements and G is the finite group acting on a Deligne-Lusztig curve X over k (or a normal subgroup G_0 of G).

In this section, we will consider the local systems on $\mathbb{G}_m/k$ that we obtain by composing f with a homomorphism

$$g : G \to M,$$

where M is a connected, simple complex Lie group. Consider the adjoint representation of M on its Lie algebra. The composite homomorphism

$$\pi_1(\mathbb{G}_m/k) \xrightarrow[f]{} G \xrightarrow[g]{} M \longrightarrow GL(\mathrm{Lie}(M))$$

defines a complex orthogonal representation Ad of the Galois group of the global field $K = k(t)$, which is unramified outside $t = 0$ and $t = \infty$. We say the homomorphism g is *rigid* if the Artin-L-function $L(\mathrm{Ad}, s)$ is identically equal to 1.

If we identify the complex numbers with an algebraic closure of $\mathbb{Q}_\ell$, the representation Ad of $\pi_1(\mathbb{G}_m/k)$ gives a lisse λ-adic sheaf $\mathcal{F}$ of rank equal to $\dim(M)$ on $\mathbb{G}_m$. Let $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ be the inclusion. Then the middle extension $j_*\mathcal{F}$, whose first cohomology gives the image of cohomology with compact supports in the cohomology of $\mathcal{F}$ over $\mathbb{G}_m$, is pure of weight 0 on $\mathbb{P}^1$. By Grothendieck's cohomological formula for $L(\mathrm{Ad}, s) = L(j_*\mathcal{F}, s)$, the adjoint L-function is constant if and only if the ℓ-adic cohomology groups of $j_*\mathcal{F}$ all vanish: $H^i(\mathbb{P}^1, j_*\mathcal{F}) = 0$ for $i = 0, 1, 2$. This implies that the M-local system on $\mathbb{G}_m$ has no deformations preserving the singularities at $t = 0, \infty$. Therefore rigid maps g give rise to rigid M-local systems on $\mathbb{G}_m$.

First, from the determination of $L(V, s)$ for irreducible representations V of G in §1, we obtain

Proposition. *The homomorphism $g : G \to M$ is rigid if and only if*

1) $\mathrm{Ad}^G = 0$

2) *the restriction of Ad to G does not contain any of the $(d-2)$ unipotent cuspidal representations which appear in $H^1(X)$.*

When $G = PGL_2(q)$ or $PSL_2(q)$, we have $d = 2$ and there are no unipotent cuspidal representations. Hence the only condition for rigidity is that $\mathrm{Ad}^G = 0$, or equivalently that the centralizer of the image of G in M is finite. We now describe some rigid embeddings in this case, by appealing to the work of Serre [S] and Greiss-Ryba[G-Ry] on finite subgroups of Lie groups.

Proposition. 1) *Assume that $G = PGL_2(q)$ and that M is of adjoint type, with Coxeter number $h = q - 1$. Then there are rigid homomorphism*

$$g : G \to M.$$

2) *Assume $q \equiv 1 \pmod 2$, that $G = PSL_2(q)$, and that M is of adjoint*

type, with Coxeter number $h = \frac{q-1}{2}$. Then there are rigid homomorphisms

$$g : G \to M.$$

Proof. If q is a prime number p, Serre [S] has constructed are principal embeddings:

$$g : PGL_2(p) \to M \qquad p = h + 1,$$

$$g : PSL_2(p) \to M \qquad p = 2h + 1.$$

These satisfy $\mathrm{Ad}^G = 0$. The only case where $q \neq p$ in the exceptional series is for M of type F_4 and E_6, where $h = 12$ and $2h+1 = 25 = 5^2$. In this case, a rigid embedding

$$g : PSL_2(25) \to M$$

has been constructed by [G-Ry].

Hence, we may assume that M is an adjoint group of classical type A, B, C, D. We tabulate M, its Coxeter number h, and a central covering $\widetilde{M} \to M$ which is a connected classical group: $\widetilde{M} \subset GL(V)$.

M	h	$\widetilde{M}$
PGL_n	n	GL_n
PSp_{2n}	$2n$	Sp_{2n}
SO_{2n+1}	$2n$	SO_{2n+1}
PSO_{2n+2}	$2n$	SO_{2n+2}

We will construct a rigid homomorphism $g : G \to M$ by constructing a complex representation V of G (or a cover $2.G$) with the appropriate dimension and self-duality.

Assume that $q = h + 1$. If $M = PGL_n$, we let V be an irreducible representation of $PGL_2(q)$ in the discrete series. These representations are all of dimension $q - 1$, so give

$$G \to GL_{q-1} \to PGL_{q-1} = PGL_n = M$$

. Since $Ad = V \otimes V^* - 1$, we have $Ad^G = 0$.

In the other cases, $h = 2n$ is even and $q = h + 1 = 2n + 1$ is odd. For $M = SO_{2n+1}$, we let S be the Steinberg representation of $G = PGL_2(q)$, which is orthogonal of dimension $q = 2n + 1$. The determinant $\epsilon = \det(S)$ is the nontrivial quadratic character of G, so the representation $V = S \otimes \epsilon$ gives a homomorphism

$$G \to SO_q = SO_{2n+1} = M.$$

In this case, $Ad = \wedge^2\ V$, so the irreducibility forces $Ad^G = 0$. We could also take $V = W \oplus \epsilon$, where W is an (orthogonal) discrete series representation of dimension $q - 1 = 2n$ and determinant ϵ. Then $\mathbb{A}d = \wedge^2\ W \oplus (W \otimes \epsilon)$ also has no G-invariants.

For $M = PSO_{2n+2}$ we can take either $V = St \oplus \epsilon$ or $V = W \oplus \epsilon \oplus 1$. Both are orthogonal representations of dimension $2n + 2$ and determinant 1, so give

$$g : G \to SO_{2n+2} \to PSO_{2n+2} = M$$

with $Ad^G = (\wedge^2 V)^G = 0$.

Finally, for $M = PSp_{2n}$ we take V an irreducible representation of dimension $2n = q - 1$ of a double cover $\widetilde{G} = 2.G = 2.PSL_2(q).2$ which is isoclinic to the group $GL_2(q)/\left\{\begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}\right\}$. The representation V has a real character, but is obstructed. It gives a symplectic representation:

$$\begin{array}{ccc} \widetilde{G} & \longrightarrow & Sp_{2n} \\ \big\downarrow & & \big\downarrow \\ G & \longrightarrow & PSp_{2n} \end{array}$$

The case when $q = 2h + 1$ is similar. Here we use the half-discrete series of dimension $\frac{q-1}{2}$ for $G = PSL_2(q)$ in the case where $\widetilde{M} = Sp$, and the half-principal series of dimension $\frac{q+1}{2}$ for $G = PSL_2(q)$ in the cases where $\widetilde{M} = SO$. We leave the details to the reader.

Note. In contrast to the case for $G = PGL_2(q)$ when $q = h + 1$, where there are many rigid embeddings, when $G = PSL_2(q)$ and $q = 2h+1$ we can only construct *two* rigid embeddings, into M, which are equivalent under the outer action of $PGL_2(q)$.

Corollary. *For the maps $g : G \to M$ described above, when $q = h + 1$ and $q = 2h + 1$, the restriction of $g \circ f$ to the decomposition group D_∞ in* $\mathrm{Gal}(K^s/K)$ *is a simple wild parameter for the split, simply-connected group over the local field $K_\infty = k((1/t))$ which is dual to M.*

Proof. We must check the two conditions [G-Re,§5.6]:

$$\mathrm{Ad}^{D_\infty} = 0$$

$$sw_\infty(\mathrm{Ad}) = \ell = \mathrm{rank}(M).$$

The first follows from the fact that the Steinberg representation St does not appear in Ad. In the exceptional cases, this follows from the work of Serre [S] and Greiss-Ryba [G-Ry], and in the classical cases from a determination of the decomposition of S^2V and $\wedge^2 V$. We then have the formula

$$sw_\infty(\mathrm{Ad}) = \dim \mathrm{Ad}^T.$$

But T acts regularly on the discrete series representations, and this allows us to calculate $\dim \mathrm{Ad}^T = \ell$.

We now consider rigid embeddings of the groups $G = PU_3(q_0)$ and $G = PSU_3(q_0)$. The group $G = PU_3(q_0)$ has a unique unipotent cuspidal representation V of dimension $2n = q_0(q_0-1)$. This representation has rational character, but is obstructed by the rational quaternion algebra ramified at ∞ and p (cf. [G]). It therefore gives a homomorphism

$$g : G \to M = Sp(V) \hookrightarrow SL(V).$$

Proposition. 1) *The restriction of $g \circ f$ to the decomposition group D_∞ in* $\mathrm{Gal}(K^s/K)$ *is a simple wild parameter for both groups SO_{2n+1} and PGL_{2n} over $K_\infty = \mathbb{F}_q((1/t))$.*

2) *The homomorphism g is rigid.*

Proof. The restriction of V to the subgroup $D_\infty = B = q_0^{1+2}(q_0^2 - 1)$ is irreducible, so there are no B-invariants in either S^2V or in $(\wedge^2 V)_0$. We have also seen that $sw_\infty(V) = 1$ and that T acts as the regular representation on $V \oplus 1$.

We now show that

$$sw_\infty(S^2V) = \dim(S^2V)^T = n$$

$$sw_\infty(\wedge^2 V_0) = \dim(\wedge^2 V_0)^T = n - 1$$

which together imply that

$$L(S^2V, t) = L(\wedge^2 V)_0, t) = 1.$$

The T-invariants in S^2V and in $(\wedge^2 V)_0$ follow immediately from the determination of the characters of T on V. To calculate the Swan conductor, we observe that the group $B = q_0^{1+2}(q_0^2 - 1)$ has irreducible representations of dimensions $q_0^2 - q_0$, $q_0^2 - 1$, and 1, with Swan conductors 1, 1, and 0, respectively. The representations of dimension $q_0^2 - q_0$ have nontrivial central characters on q_0^{1+2}, and the representation of dimension $q_0^2 - 1$ has trivial central character, but is nontrivial on q_0^{1+2}. From this, the Swan conductors of S^2V and $(\wedge^2 V)_0$ are easily determined.

Since the adjoint representations are

$$\mathrm{Ad} = S^2V \qquad \text{for } \ M = Sp(V)$$

$$\mathrm{Ad} = S^2V + (\wedge^2 V)_0 \qquad \text{for } \ M = SL(V)$$

We see that $g \circ f$ is a simple wild parameter for the groups SO_{2n+1} and PGL_2 over K_∞, and that the homomorphism g is rigid. This completes the proof.

Note. When $q_0 = 8$, we have $2n = q_0^2 - q_0 = 56$. In this case, the subgroup $G_0 = PSU_3(8)$ of index 3 in G stabilizes a symmetric quartic form on V, and we obtain an embedding [G-Ry]:

$$g : PSU_3(8) \to E_7 \subset Sp(V)$$

which is also rigid. The restriction of

$$\pi_1(\mathbb{G}_m/\mathbb{F}_{64}) \xrightarrow[f]{} PSU_3(8) \xrightarrow[g]{} E_7$$

to $D_\infty = 8^{1+2}.21 = 2^{3+6}.21$ is a simple wild parameter for E_7 over $\mathbb{F}_{64}((1/t))$. We have:

$$sw_\infty(V) = 3 \qquad \deg\ L(V,t) = 1$$

$$sw_\infty(\mathrm{Ad}) = 7 \qquad \deg\ L(\mathrm{Ad},t) = 0.$$

We have now constructed all of Katz's local systems for G_2, with the exception of the group $SL_2(8)$ in char 2. The example of $SL_2(8)$ fits into a series of embeddings [S, pg. 422]

$SL_2(4)$	in	$SO_3 = A_1^{ad}$
$SL_2(8)$	in	G_2
$SL_2(16)$	in	D_8^{ad}
$SL_2(32)$	in	E_8
$PSL_2(9)$	in	A_2^{ad}
$PSL_2(27)$	in	F_4

where the image of the Sylow p-subgroup is not contained in a maximal torus. We leave it to the reader to check that these embeddings are all rigid (the first four in char 2 and the last two in char 3). In each case, the restriction to D_∞ is a simple wild parameter.

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